FHLT Talk 3. B. Morrissey.

1 pt, line ~~&~~ surface ~~&~~ ops.

<u>Example</u> 't-Hooft ops : Wilson.

$$Z(X) = \int_{\mathcal{M}} e^{2\pi i \mathcal{L}(x)} \, d\mu.$$

$Z'(X)$ replace $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) \cdot \int_{\gamma} \mathrm{tr}_A(A)$

the of flat conn. contour $A \in \mathrm{Rep}(G)$.

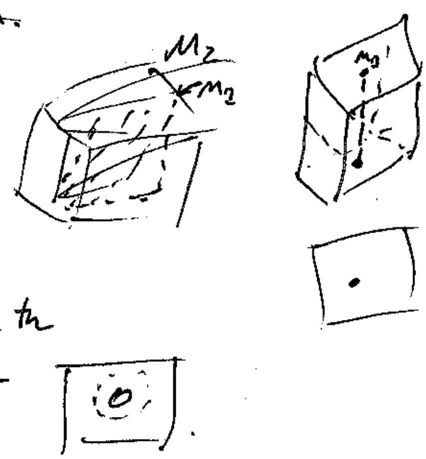
~~S~~ We wish to generalize our idea of a TFT (Law) to one allowing labelings of certain submanifolds.

~~Defn (J. Lurie) Singularity datum.~~

<u>Easy case</u> : $M_1 \subset M_2$.

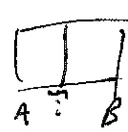
take perpendicular slice.

Consider evaluating bordism homology with some special value to ⊙ ⤳. Then this is the same as if we had a bdy



$\Rightarrow Z(S^1)$ classifies labellings.

Differences : Domain walls.



(A,B) - bimodule.

Working towards ... Cobordism hypothesis w/ singularities §4.3 of Lurie.

1. Singularity datum.

a). Recall (X, ξ) mflds. $M \xrightarrow{p} X$, with $\xi \downarrow X$, and $p^*\xi \downarrow M$ (stably ≅ trivialization).

Defn (inductive). Singularity data of length k, dim n.
of length 0: (X, ξ).

Sig data of length k:

$$(\vec{X},\ X_k,\ \xi_k,\ p: E_k \to X_k).$$

$\vec{X}$ ⇒ n-di sig data of length $k-1$.
X_k - top space.
ξ_k - real vect bdle of dim $n-k$ of X_k w/ inner product.
$p: E_k \to X_k$ fibre bdle, fibre on each $x \in X_k$ is get $\vec{X}$ mfld, of codim $\xi_x \oplus \mathbb{R}$.

$(\vec{X}', X_k, \xi_k, p: E_k \to X_k)$.
$\vec{X}'$ mfld of codimens V. (V real vect space dim $m \leq n-k$)

(i) top space M.
(ii) subspace $M_k \subset M$. such mflld dim $n-m-k$, has tgt bdle T.
(iii) $q: M_k \to X_k$, $T \oplus \underline{V} = q^* \xi$
($\underline{V}$ const v.b. associated to V).
Then make $q^*E = E \times_{X_k} M_k$ a $\vec{X}'$ mfld of codim $V \oplus \mathbb{R}$. ⇒ $q^*E \times (0,1)$ a $\vec{X}'$-mfld of codim V.
iv) $\vec{X}'$ mfld struct on $M - M_k \subset M$.
v) open neighborhood U of M_k, map $f: (0,1] \times q^*E \to U$
$f|_{(0,1) \times q^*E}$ is open embedding of $\vec{X}'$ mflds.
$f|_{\{1\} \times q^*E} = \begin{pmatrix} q^*E \\ \downarrow \\ M_k \end{pmatrix}$

$\vec{X}$ mfld of dim m, ⇒ $\vec{X}$ mfld codim $\mathbb{R}^{n-m}$.

Rmk: $\{X_i\}_{0 \le i \le k}$. v.bs $\{\xi_i\}_{0 \le i \le k}$

fibre bdls $\{E_i \to X_i\}_{0 \le i \le k}$.

A mfld corresponds to

$$M_n \subseteq M_{n-1} \subseteq M_{n-2} \subseteq \dots \subseteq M_0 = M.$$

$M_k \setminus M_{n+1}$ smth dim $n-k$. with (X_k, ξ_k) structure.

$p_i : E_i \to X_i$ describs fit.

$$\begin{array}{ccc} \mathrm{Bord}_n^{(X,\xi)} & \hookrightarrow & \mathrm{Bord}_n^{\vec{X}} \\ Z_0 \downarrow & \swarrow & \\ \mathcal{C} & & \end{array}$$

(all M_i's empty).

Q. What is needed lifty data.

Thm: Cobordism hypothesis with singularities:

$\vec{X} = (\vec{X}', X_k, \xi, p : E \to X)$.

sg data length k.

$\tilde{X}$ bdle of orthonormal frames in ξ.

$\downarrow$

X

(ppl $O(n-k)$ bdle over X.)

For $\tilde{x} = (x, \alpha : \xi_x \simeq \mathbb{R}^{n-k})$ we can use α to view fiber $p^{-1}(x)$ as a $\vec{X}'$ mfld of codim $\mathbb{R}^{n+1-k}$, which determ an object $E_{\tilde{x}} \to \Omega^{k-1}\mathrm{Bord}_n^{\vec{X}'}$

($\tilde{x} \rightsquigarrow E_{\tilde{x}}$ eqvt w.r.t. $O(n-k)$).

Why $Z : \mathrm{Bord}_n^{\vec{X}}$ equiv to:

- $Z_0 : \mathrm{Bord}_n^{\vec{X}'} \to \mathcal{C}$.
- family of 1-morphisms $\eta_{\tilde{x}} : 1 \to Z_0(E_{\tilde{x}})$ in $\Omega^{k-1}\mathcal{C}$.

$\tilde{x} \to \eta_{\tilde{x}}$ is $O(n-k)$ equivariant.

Prf.: Omitted.

Cobordism hypothesis with singularities:

examples: (Feynman diagrams, loop ops. domain walls).

a). Feynman diagrams.

$\vec{X}$ 1 d, length 1 sig data

X_0, X_1. ζ_0 on X_0 rk 1 vect bdle.

$\tilde{X}_0$ = assoc. double cover of X_0.

Corr. space $E \to X_1$ finite fibres, cts. map $E \to \tilde{X}_0$.

Assume $\pi_i \tilde{X}_0$ vanish $i > 0$.

particles mapped to X_0 – classified by $\pi_0 X_0$. (double cover means antiparticles exist)

interactions mapped to X_1.

The neighbourhood of $E_x = E \times_{X_1} \{x\}$

= finite set, with map

$\sigma_x : E_x \to P$.

$\vec{X}$ mfld.

top space M, smooth away from

finite set $M_0 \in M$.

$M \setminus M_0$ – labelled by particles

M_0 – labelled by interactions. $\rightsquigarrow$

E_x gives the edges that meet it.

$(E_x \times (0,1]) \coprod_{E_x \times \{1\}} \{\text{pt}\}$.

$ob(Bord_1^{\vec{X}})$ = particles

$mor(Bord_1^{\vec{X}})$ = Feynman diagrams.

Thm 4.3.11: • ~~Vect~~ TFT gives

V_p to each $p \in$ Particles. $V_p \otimes V_{\bar{p}} \to k$.

vect $v_x \in \bigotimes_{e \in E_x} V_{\sigma(e)}$ for each interaction.

This gives obv. lin map ~~of~~ between bordisms,

Example 2: codim k operators.

$X_0 = BO(n)$.

$X_1 = \emptyset$

$\vdots$

$X_k = BO(n-k)$.

ζ_0, ζ_k are tautological bdles.

~~$E_k = (X_0 \downarrow X_k)$~~ $E_k = X_k \times S^{n-k-1}$

Cob hyp: We need ~~element of~~ map $1 \to Z(S^{n-k-1})$. / element of $Z(S^{n-k-1})$.

Example 3: Domain walls.

$X_0 = BO(n) \amalg BO(n)$. ζ_0, ζ_1 tautological.

$X_1 = BO(n-1)$.

$E_1 = X_1 \times \{x, y\}$.

A mfld is $M \simeq M_- \amalg_{M_0} M_+$.

Cob hyp: • A pair of objects $C, D \in \mathcal{C}$. htpy fixed pt for $O(n)$-action on $\mathcal{C}^\sim$.

[determine TFT's Z_+, Z_- on M_+, M_- respectively].

• Maps $1 \to$ ~~$C \otimes D$~~ $C \otimes D$. equivt. wrt $O(n-1)$ action.

[note $C \simeq C^\vee$ $O(n-1)$ equivt so equiv to a map $C \to D$].

FHLT HR ~~43~~. pg 4 Benedict Morrissey

Examples of ~~[illegible]~~ operators.

1. Dijkgraaf — Witten theories in dimension 2 & 3.

dimension 2. : Fully dualizable object $A = \mathbb{C}^{\tau}[G] \in \underbrace{Alg^1(Vect)}_{\text{2 category}}$.

pt-like operators $= Z(S^1) = A \otimes_{A \otimes A^{op}} A = A/[A,A]$.

(When $A = \mathbb{C}^{\tau}[G]$, $\tau = 0$ this is $Cl[G]$ = class fns)

Line operators : (A, A) - bimodules.

These have an non-commutative algebra structure.

$Z(S^1) \to Z(S^1)$ given by

$l_1 \otimes l_2 \otimes l_3 \to l_{prod}$.

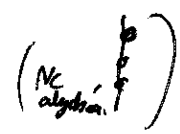
may line ops 'together'. & then is a braiding.

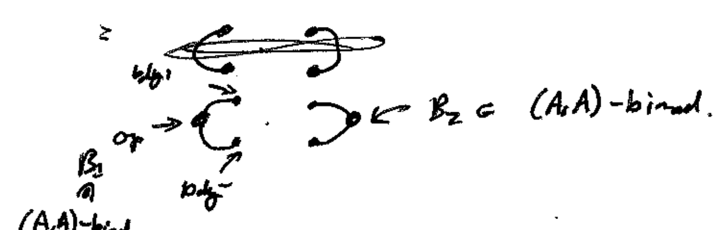
pt-like operators on a line

Here $X_0 = BO(2)$, $X_1 = BO(1)$, $X_2 = BO(0)$.

$E_1 = X_1 \times \{a, b\}$.

$E_2 = X_2 \times$  (without line op).





So get $B_1 \overset{L}{\otimes}_{A \otimes A^{op}} B_2$.

(Elements should give morphism from B_1 to B_2 in (A,A)-bimod. ~~Q: [illegible] with this?~~ Ex try do.)

Since $(B_1 \overset{L}{\otimes}_{A \otimes A^{op}} B_2) \otimes (B_2 \otimes B_3) \to B_1 \otimes B_3$.

DW theory in dimension 3.

Recall $A = Z(pt) = \mathrm{Vect}^\tau[G] \in \mathrm{Mor}_3$. {objs (X, ε_X) $\varepsilon_X(-): X \otimes - \to - \otimes X$ natural transform. compatible w/ monoidal structure}.

$Z(S^1) = \underline{Z}(\mathrm{Vect}^\tau[G]) = A \otimes_{A \otimes A^{op}} A$
(Drinfeld Center)

$\cong \mathrm{Hom}_{A \otimes A^{op}}(A, A)$

because $A \simeq A^\vee$ from bilin form corresponding $A \otimes A \to \mathrm{Vect}$.
$W \otimes W' \mapsto \theta(W * W')$
$\theta: A \to \mathrm{Vect}$, taks fiber on identity elmt.

= {twisted equivt v.b.'s $W \to G$ (with twist lifts $L_{x,y} \otimes W_x \to W_{yxy^{-1}}$ of G-action by conjugation, $L \to G \times G$ defined by $K^*_{yxy^{-1}, y} \otimes K_{y,x}$.} (*)

Prop 4.9 of FHLT

Components we hand.

{$W \to G$ & Z center).
$W \to G$ defined by
C_y at $y \in G$, 0 elsewhere.
braiding gives $\forall x$ ch.
$K_{y,x} \otimes C_y \otimes W_x \to K_{yxy^{-1}, y} \otimes W_{yxy^{-1}} \otimes C_y$. which is (*)
⇒ necessary. Q. why sufficient?

= Line operators.

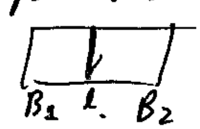
Surface operators = $(\mathrm{Vect}^\tau[G], \mathrm{Vect}^\tau[G])$ bimodule.

Note: on $S^2 \times [0,1]$ this has braiding & product.

* So we just attach vect space to each pt!

Note: B_1, B_2 surface $\in$ ~~det~~ (A, A)-bimod.

$B_1 \otimes_{A \otimes A^{op}} B_2 \rightsquigarrow$ line ops ~~on a~~



FHLT HK #3. pg 5. B. Morrissey

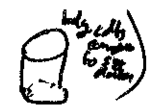
DW thy in dimension 3. Pg 2.

pt operates $= Z(S^2) \cong \Gamma(\mathrm{Hom}(S^2, BG), \mathcal{L}_\kappa)$.

by the fact this is DW thy ~~ind~~ on $(G, \mathbb{Z}/\pi_1)$ bundles by identified ~~action~~ (3,2,1) with Freed - Quinn finite Chern-Simons.

pt operats on a line

$= Z($  $)$

$= Z($  $)$

bdy cells ...

= Functs between two (A,A) bimodules

Line op on a surface:

$= Z($  $)$

$= Z($  $) \cdot Z($  $)$

cone bdy other

Same arr.

$\mathrm{Vect}^{\tau_i}\mathrm{ev}[G]$,

disc gives maps $\to 1$.

$= \mathrm{Maps}(1 \to \mathrm{Vect}^{\tau_i \mathrm{ev}}[G] \to 1)$.

This is as far as I am going.

* Add ~~Betty~~ Domain Walls as Tilting Modules.

How does.

FHLT Tlk3 pg 46 B. Morrissey

Doing this for the Kapustin – Thorngren TFT.

Idea replace BG with $B\mathcal{G}$.

~~Ex~~ We have data $(\Pi_1, \Pi_2, \alpha : \Pi_1 \to Aut(\Pi_2), \beta \in H^3(B\Pi_1, \Pi_2))$.

or (G, H, t, α). $t: H \to G$, $H =$ 2-morphisms ~~with~~ Id → something. t: 2-morphism → Image }.

$\alpha : G \to Aut(H)$. [By cjgtn].

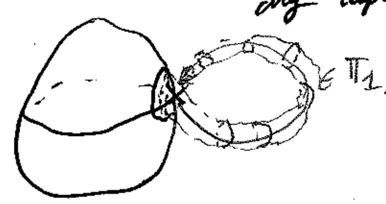
Postnikov tower type constructions. $\Pi_2 = Ker(t)$, $\Pi_1 = coker(t)$. β = gen class of $\Pi_2 \to H \to G \to \Pi_1$.

$B\Pi_2 \longrightarrow B\mathcal{G}$
$\downarrow$
$B\Pi_1$

Such extensions are classified by $\beta \in H^3(B\Pi_1, \Pi_2)$.

We get a nondegenerate action $\alpha : \Pi_1 \to Aut(\Pi_2)$

Take open set and pt, structure it only loop.



Get new element of Π_2!

Claim: (partly FHLT. part mine).

Rep$[\Pi_2][\Pi_1]$ is (untwisted version).

category ⇝ to each pt of Π_1 we associate a representation of Π_2. ~~&~~ → with same type of operator but here K is a Rep(Π_2) bdle. (sd?)

This is a fusion category & hence fully dualizable in Mor_3. [by Douglas – Schommer-Pries – Snyder].

* Open Q: What are these classified by.

Defn: A fusion category is a rigid (left & right dualizable objects) semisimple linear monoidal cat, with only finitely many isomorphism classes of simple objects, $End(\mathbb{1}) \cong k$.

Note: Fusion cats are representation cats of weak Hopf algebras.

P.T.O.

Firstly: Why is this reasonable:

$$Z(S^1) = A \otimes_{A \otimes A^{op}} A \cong \mathrm{Hom}_{A \otimes A^{op}}(A, A) \cong Z_{Drin}(A).$$

and $A \cong A^\vee$ by

$$\theta : A \otimes A \to \mathbb{C}.$$
$$w_1 \otimes w_2 \longmapsto \theta(w_1 * w_2).$$
$$= tr((W_1 * W_2)_{Id\pi_2}).$$

~~$Z(S^1) =$~~ Letting $W' \to G$ be the π_2 rep at $y \in G$, Otherwise we get via the braiding a map $K_{y,x} \otimes W_x \to K_{yxy^{-1}, y} \otimes W_{yxy^{-1}}$.

So $Z_{Drin}(A) =$ "Exact" π_2-rep bundles.

<u>Claim:</u> This is a modular tensor category

Hence by the Reshetikhin-Turaev thm is a 3-2-1 thy with the value of $Z_{RT}(S^1) = Z_{Drin}(A)$.

~~Claim~~

<u>Conjecture:</u> These 4D thy extends the RT thy.

$Z_{RT}(M^3) =$

Problem: the definition ~~requires~~ uses surgery of 3 mflds. → unclear (to me!) how to calculate. → Can try & and instead in the Mor$_3$ - but again - in final Z levels can't use factorization homology.
→ Can try & use FHLT §8. → same problem as last time.

<u>Operators:</u> $Z(S^1)$ = Line ops. $Z(S^2)$ & others same as in prev section.

Anomalies. (really of Reshetikhin Turaev).

1. Chern-Simons anomaly:

There is a term ? that is dependent on a trivialization of tangent bundle.

If we have two different trivializations, differ by an integer. $I(g) \to I(g) + 2\pi s$.

One way to provide such a trivialization is by making it the bdry of a 4 mfld.

Q. Why??

ie. Extra data is needed in some way -

we can modify the integral by some ~~...~~ an integral contribution over the 4-mfld.

2. Interpretation as $1 \xrightarrow{Z_{CS}} Z_{Sky}$. in an approp 4-category.

Some can see as domain walls / bdry between a TFT & the trivial TFT.

4. (follow §5, 6, 9 of FHLT).

Let us try to modify the description of the 3d extended DW thy $\rightsquigarrow$ which gave the Chern-Simons thy for a finite gauge group, for a torus T.

Other cases of anomalies in physics involve ~~terms~~ thys living in twisted bdls, cancelling anomalies terms with another bdle to give const-trivial.

- We are again get $H^4(BT, \mathbb{Z})$ to classify hermitian line bdls

$$\begin{array}{c} K \\ \downarrow \\ T \times T \end{array}$$ with $\theta_{x,y,z}$'s as ~~...~~ in DW case.

$Sky^{\tau}[T] = \{$objects: Skyscraper sheaves, finite support, stalks in Vect$\}$.

[so that convolution still works].

Problem: This is not fully dualizable.

* Why?

we can evaluate on the circle:

$$Z(S^1) = Z_{?}(Skg^{\tau}[T]) \simeq Sky^{\tau}[t] \otimes Sky^{\tau}[\hat{F}]$$

$$t \times \hat{F} = (t \times \Lambda)/\Pi \qquad (T = t/\Pi)$$

$\Lambda = Hom(T, \mathbb{T}) = H^1(T, \mathbb{Z})$.

$\Pi = Hom(\mathbb{T}, T) = H_1(T, \mathbb{Z})$. $\quad \mathbb{T} = \mathbb{C}^\times$.

∃ Morita equivalence:

$Vect \sim Sky^{\tau}[t]$ gives $A_t \to \mathbb{1}$.

$Z_{cs}(M) \in Z_A(M)$-Mod.

$\Rightarrow \; 1 \xrightarrow{Z_{cs}(pt)} Z_A(pt)$. is a morphism.

Chern Simons: $Z \; Sky^{\tau}[T]$ gives bimodule $A_F \to A_t$.

where A_F, A_t given by $Sky^{\tau}[t]$, $Sky^{\tau}[\hat{F}]$

~~except~~ because $Sky^{\tau}[T]$ is a $(Sky^{\tau}[t], Sky^{\tau}[\hat{F}])$-bimod. (only need braided)

(prop 6.5).

$Sky^{\tau}[t]$, $Sky^{\tau}[\hat{F}]$ are ribbon categories which are E_2-algebra objects in Cat. ⇒ Define a 4d not 3d TFT.

Note $Z_{A_F}(S^1)$ = Braided bimodule cats over $Sky^{\tau}[F]$ which is where (together w/ t part) Z_{cs} cat lives).

* There is also a ... approx

$t_{(n)} \subset t$ discrete lattice — similar story & Maurer eqs.

FHLT talk 3. pg 8. D. Morrison

From FHLT §9:

dim X.	$\mathcal{A}_t(X)$	$Z(X)$	$\mathcal{A}_F(X)$
0	$Sky^{\tau}[t]$	$Sky^{\tau}[T]$	$Sky^{\tau}[\hat{F}]$
1	$t \ltimes_b Sky[t^*]$	$Sky^{\tau}[t] \otimes Sky^{\tau}[F]$	$F \ltimes_b Sky[F^*]$
2	$W_{\tau}(H^2(X, t))$	$L^2(J_T(X), \Theta(\tau))$	$W_{\tau}(H^2(X, F))$
3	$\mathbb{C}$	$Z(X)$	$\mathbb{C}$
4	$\mathcal{A}_t(X)$ a sm dstn elm		$\mathcal{A}_F(X)$.

$W(V)$ = Weyl algebra.

$L^2(J_{\tau}(X),$

* Actually can't really explain.

$L^2(J_{\tau}(X), \Theta(\tau))$.

Just

---

* Mattel's phone is charging @ back ⟺ if anyone needs a new phone.