Factorization Homology

Following Ayala-Francis: Factorization Homology of Topological Mfds, 2015.

Conventions: • $\mathbb{K}$: field, char 0.

• All categories are ∞-categories; when I describe a top. enriched 1-category, take coherent nerve to get ∞-cat.

<u>Disk Algebras</u> (3.1 in AF 15)

Mfd_n : category with objects n-mfds, $\mathrm{Mfd}_n(M,N) := \mathrm{Emb}(M,N)$, compact-open topology.

Disk_n : full subcat. gen. by $\mathrm{Mfd}_n \ni U \cong \coprod_I \mathbb{R}^n$.

<u>Def</u> A <u>Disk_n-algebra</u> A is a symmetric monoidal functor:

$$\cancel{A \in \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n, \mathrm{Chain}_{\mathbb{K}})}$$

$$A \in \mathrm{Fun}^{\otimes}((\mathrm{Disk}_n, \amalg), (\mathrm{Chain}_{\mathbb{K}}, \otimes)).$$

<u>E.g.</u> ① $(\mathrm{Disk}_n, \amalg) \longrightarrow (\mathrm{FinSet}, \amalg) \xrightarrow{F} (\mathrm{Chain}_{\mathbb{K}}, \otimes)$

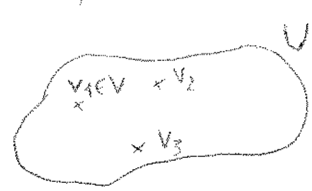
$$U \cong \coprod_I \mathbb{R}^n \longmapsto I$$

$\cancel{B := F(1)}$

$$F(2) = B \otimes B \xrightarrow{\text{mult.}} B := F(1) \xleftarrow{\text{unit}} F(0) = \mathbb{K}$$

② Fix $V \in \mathrm{Chain}_{\mathbb{K}}$, the free Disk_n-alg. generated by V is:

$$\cancel{F(V)}\ \mathrm{Free}(V)(U) := \bigoplus_{j \geq 0} C_*(\mathrm{Conf}_j(U)) \underset{\Sigma_j}{\otimes} V^{\otimes j}$$



3. "loop spaces" $\quad A(U) = \mathrm{Ch}^{\bullet}_{*}(\mathrm{Map}_c(U,X))$

Fix $X \in \mathrm{Top}$.

$$\mathrm{Map}_c(\mathbb{R}^n, X) \simeq \mathrm{Map}_*(S^n, X) = \Omega^n X$$

Factorization Homology

Def $A \in \mathrm{Disk}_n\text{-}\mathrm{Alg}$, $M \in \mathrm{Mfd}_n^B$. Factorization homology of M with coeffs in A is:

$$\int_M A := (\mathrm{ho})\mathrm{colim}\left(\mathrm{Disk}^B_{n/M} \longrightarrow \mathrm{Disk}^B_n \xrightarrow{A} \mathrm{Chain}_{\mathbb{k}}\right).$$

$$U, V \in \mathrm{Disk}^B_{n/M}, \quad U \cong \coprod_I \mathbb{R}^n, \quad V \cong \coprod_J \mathbb{R}^n$$

$$\begin{tikzcd}
\cdots & A(U) \cong A^{\otimes I} \arrow[l] \arrow[r, shift left] \arrow[dr] & A(V) \cong A^{\otimes J} \arrow[l, shift left] \arrow[r] \arrow[d] & \cdots \arrow[dl] \\
& & \int_M A &
\end{tikzcd}$$

Note: If $M \in \mathrm{Disk}^B_n$, all embs. $\{U \to M\}_{U \in \mathrm{Disk}^B_n}$ factor through $M \xrightarrow{id} M$,

so $\quad \mathrm{colim} = A(M)$.

Note: This colimit is a left Kan extension.

E.g. 1. $A = \mathrm{Sym}(V)$, for $V \in \mathrm{Ch}_{\mathbb{k}}$. Then $\int_M A = \mathrm{Sym}(\mathrm{Ch}_*(M) \otimes V)$.

2. $A = \mathrm{Free}(V)$. Then $\int_M A = \bigoplus_{j \geq 0} \mathrm{Conf}_j(M) \otimes_{\Sigma_j} V^{\otimes j}$.

Note: not htpy invariant.

3. $A(U) = \mathrm{Ch}^{\bullet}(\mathrm{Map}_c(U,X))$. Then $\int_M A = \mathrm{Ch}^{\bullet}(\mathrm{Map}_c(M,X))$.

"Non-abelian PD". $H^{\bullet}(\mathrm{Map}_c(M,X))$ is impossible to compute, in general, using classical techniques. Important example: M compact, $X = BG$, then $\mathrm{Map}_c(M, BG) = \mathrm{Bun}_G(M)$.

Closed unit interval (3.2 in AF15)

$\exists$ variant $\mathrm{Disk}_n^{\partial,\mathrm{or}}$. In particular, look at $\mathrm{Disk}^{\partial,\mathrm{or}}_{1/[0,1]}$

$$\mathrm{Fun}^{\otimes}(\mathrm{Disk}^{\partial,\mathrm{or}}_{1/[0,1]}, \mathrm{Chain}_{\mathbb{k}}) \cong \mathrm{Alg}_{RL}(\mathrm{Ch}_{\mathbb{k}})$$

$$\Psi$$

$$(F[0,1), F(0,1), F(0,1]) \longleftrightarrow (R, A, L):$$

R is right A-module
A is a $\mathbb{k}$-algebra (DGA)
L is left A-module

$F(0,1)$ is a DGA: $(0,1) \amalg (0,1) \hookrightarrow (0,1)$

$$F(0,1) \otimes F(0,1) \xrightarrow{\text{mult.}} F(0,1)$$

$F[0,1)$ is a right $F(0,1)$-module:

$$[0,1) \amalg (0,1) \hookrightarrow [0,1)$$
$$F[0,1) \otimes F(0,1) \to F[0,1)$$

Let $S \subset \mathrm{Disk}^{\partial,\mathrm{or}}_{1/[0,1]}$ be full subc. consisting of

$U = [0,1) \amalg (0,1)^{\amalg J} \amalg (0,1]$, let $i: S \to \mathrm{Disk}^{\partial,\mathrm{or}}_{1/[0,1]}$ be inclusion.

<u>Lemma</u> i is final, i.e. $\forall\, F \in \mathrm{Fun}(\mathrm{Disk}^{\partial,\mathrm{or}}_{1/[0,1]}, \mathrm{Chain}_{\mathbb{k}})$,

$$\operatorname{colim} F \cong \operatorname{colim} F \circ i.$$

Pf (Lemma 3.11 in AF15.)

<u>Cor</u> $\int_{[0,1]} A$ is a 2-sided bar construction.

<u>Pf</u> $\displaystyle\int_{[0,1]} A = \operatorname*{colim}_{J \in \mathrm{FinSet}} A[0,1) \otimes A(0,1)^{\otimes J} \otimes A(0,1]$,

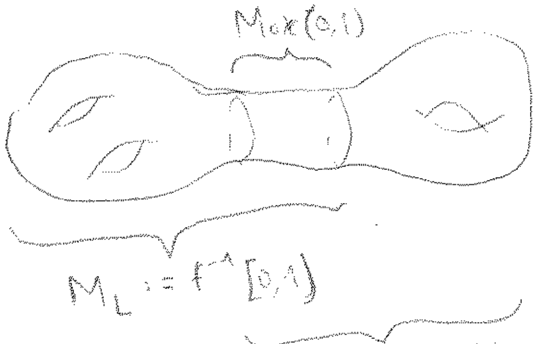
$$= A[0,1) \overset{\mathbb{L}}{\underset{A(0,1)}{\otimes}} A(0,1].$$

$$\cancel{\mathcal{F}(M_L) \otimes_{\mathcal{F}(M \times \mathbb{R})} \mathcal{F}(M_R) \overset{\mathrm{cor}}{\simeq} \int_{[-1,1]} \mathcal{F} \circ f^{-1}}$$

Homology Theories (AF15, §3.3 - §3.5).

"Any homology theory must satisfy excision."

Def A collar-gluing of B-framed mfds. is a cts. map $f : M \to [0,1]$, such that $f^{-1}(0,1) \cong (0,1) \times M_0$, for M_0 a $n-1$-mfd.



$$M = M_L \bigcup_{M_0 \times \mathbb{R}} M_R$$

Now for $\mathcal{F} \in \mathrm{Fun}^{\otimes}(\mathrm{Mfd}_{n/M}, \mathrm{Chain}_{\mathbb{K}})$,

$\mathcal{F} \circ f^{-1} \in \mathrm{Fun}^{\otimes}(\mathrm{Disk}^{\partial,\mathrm{or}}_{n/[0,1]}, \mathrm{Chain}_{\mathbb{K}})$.

$$\mathcal{F}(M_L) \overset{\mathbb{L}}{\underset{\mathcal{F}(M_0 \times \mathbb{R})}{\otimes}} \mathcal{F}(M_R) = \mathcal{F} \circ f^{-1}[0,1) \overset{\mathbb{L}}{\underset{\mathcal{F} \circ f^{-1}(0,1)}{\otimes}} \mathcal{F} \circ f^{-1}(0,1] \simeq \int_{[0,1]} \mathcal{F} \circ f^{-1}$$

$\downarrow$ univ. prop. of colimit?

$\mathcal{F}(M)$

Def We say $\mathcal{F} \in \mathrm{Fun}^{\otimes}(\mathrm{Mfd}_{n}, \mathrm{Chain}_{\mathbb{K}})$ is a homology theory if $\forall$ collar gluing $M = M_L \bigcup_{M_0 \times \mathbb{R}} M_R$

$$\int_{[0,1]} \mathcal{F} \circ f^{-1} \xrightarrow{\simeq} \mathcal{F}(M).$$

$\mathbb{H}(\mathrm{Mfd}_{n}, \mathrm{Chain}_{\mathbb{K}})$ is the full subcat. of $\mathrm{Fun}^{\otimes}$ consisting of Homology theories.

Lemma (Lemma 3.18 in AF15)

$\int A \in \mathbb{H}(\mathrm{Mfd}_n, \mathrm{Chain}_{\mathbb{K}})$.

Cor $A \in \mathrm{Fun}^{\otimes}(\mathrm{FinSet}, \mathrm{Chain}_{\mathbb{K}})$, i.e. an associative algebra,

$$\int_{S^1} A = \int_{\mathbb{R}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}} A}{\otimes}} \int_{\mathbb{R}} A = A \underset{A\otimes A^{op}}{\otimes} A = CH_\bullet(A,A).$$

HOCHSCHILD CHAINS.

Theorem (3.24 in AF15) $\int : \mathrm{Disk}^B_n\text{-Alg} \xrightarrow{\simeq} \mathbb{H}(\mathrm{Mfd}^B_n, \mathrm{Chain}_{\mathbb{K}}) : ev_{\mathbb{R}^n}$.

Pf $\int$ is a symmetric monoidal Kan extension, so it's the left adjoint in:

$$\int = i_! : \mathrm{Disk}^B_n\text{-Alg} \rightleftarrows \mathrm{Fun}^{\otimes}(\mathrm{Mfd}^B_n, \mathrm{Chain}_{\mathbb{K}}) : i^*.$$

(Here i is inclusion : $\mathrm{Disk}^B_n \to \mathrm{Mfd}^B_n$.)

Unit $1 \to i^* i_!$ is equivalence, because left Kan ext. along a fully faithful functor restricts as the original functor.

Counit $i_! i^* \to 1$ evaluates on $\mathcal{F} \in \mathrm{Fun}(\mathrm{Mfd}^B_n, \mathrm{Ch}_{\mathbb{K}})$ to $\int A \to \mathcal{F}$, where $A = \mathcal{F}(\mathbb{R}^n)$.

Remains to show $\int_M A \simeq \mathcal{F}(M)$, $\forall M \in \mathrm{Mfd}_n^B$.

1. $\int_{\mathbb{R}^n} A = \mathcal{F}(\mathbb{R}^n)$.

2. Symm. mon. $\Longrightarrow \int_{\coprod_J \mathbb{R}^n} A = \mathcal{F}\left(\coprod_J \mathbb{R}^n\right)$. suffices to show for connected mfds.

3. By induction, $\int_{S^k \times \mathbb{R}^{n-k}} A \simeq \mathcal{F}(S^k \times \mathbb{R}^{n-k})$.

   1 is base case, excision for inductive step:

$$S^k \times \mathbb{R}^{n-k} = \mathbb{R}^k \times \mathbb{R}^{n-k} \bigcup_{S^{k-1}\times\mathbb{R}^{k+1}} \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

4. Use handlebody decomp. & excision, extra care needed for $n = 4$. □

Application.

<u>Theorem</u> (Lurie, Thm 4.1.)
Let $A \in \mathrm{Disk}^{fr}_n\text{-}\mathrm{Alg}(\mathcal{C})$. Then A is fully dualizable, so CH gives a unique TFT

$$Z \in \mathrm{Fun}^{\otimes}(\mathrm{Bord}^{fr}_n, \mathcal{C})$$

such that $Z(*) = A$. Moreover, $\forall$ M an n-morphism in Bord^{fr}_n,

$$\int_M A \simeq Z(M).$$