Integrating Quantum Groups over Surfaces

0. Plan

- introduce quantum groups (Hopf algebras) and their categories of representations: braided monoidal structure.
- recall factorization homology, this time with target the 2-category of categories (presentable, k-linear with exact functors)
- E_2 algebras are braided monoidal cats; intrested mostly in $\operatorname{Rep} \mathcal{U}_q\mathfrak{g}$.
- take FH of punctured surfaces, get useful repr. theoretic invariants, such as $\operatorname{QCoh}(\underline{\operatorname{Ch}}_G(S))$. This is spectral side in Betti geometric Langlands.

1. Quantum Groups

X space $\rightsquigarrow \mathbb{C}(X) = \operatorname{Map}(X, \mathbb{C})$ algebra, using multiplication on $\mathbb{C}$

G top. group $\rightsquigarrow \mathbb{C}(G) = \operatorname{Map}(G, \mathbb{C})$ bi-algebra, using multiplication on G:

$$\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$$

Actually Hopf algebra: antipode S, $S(f)(g) = f(g^{-1})$ compatible w/ mult. and comult.

$\operatorname{Map}(X, \mathbb{C})$ always commutative, but cocommutative $\iff$ G is abelian.

Another Hopf algebra we can build out of G: universal enveloping algebra.

$$G \rightsquigarrow \mathfrak{g} \rightsquigarrow \mathcal{U}\mathfrak{g} := T(\mathfrak{g}) / (x_1x_2 - x_2x_1 - [x_1, x_2])$$

$\Delta(x) = x \otimes 1 + 1 \otimes x$

$S(x) = -x$

~~taks~~ co-commutative, ~~but isn't so on higher tensors~~.

In nice cases (G connected, 1-connected, ...)

$\mathbb{C}(G)$-comodules $\longleftrightarrow$ G-reps $\longleftrightarrow$ $\mathcal{U}\mathfrak{g}$-modules.

<u>Def</u> Quantum groups are non-commutative deformations of $\mathbb{C}(G)$, or non-cocommutative deformations of $\mathcal{U}\mathfrak{g}$, in the category of Hopf algebras.

<u>Eg</u> $\mathcal{U}_q\, sl_2(\mathbb{C})$ is the quotient of the free algebra on generators K, K^{-1}, E, F by the relations:

$$K K^{-1} = 1 = K^{-1} K$$
$$K E K^{-1} = q^2 E$$
$$K F K^{-1} = q^{-2} F$$
$$EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}}$$

How do we recover $\mathcal{U}\, sl_2\mathbb{C}$?

<u>Def</u> $\mathrm{Rep}(\mathcal{U}_q\mathfrak{g})$ is the category of (integrable? f.d.?) reps. of $\mathcal{U}_q\mathfrak{g}$ <u>as an algebra</u>.

The coalgebra structure on $\mathcal{U}_q\mathfrak{g}$ is used to define a meaningful action on $M\otimes N$, and thus get monoidal structure on $\mathrm{Rep}(\mathcal{U}_q\mathfrak{g})$.

$U := \mathcal{U}_q\mathfrak{g}$

$$\mathcal{U} \xrightarrow{\Delta} \mathcal{U}\otimes\mathcal{U} \longrightarrow \mathrm{End}(M)\otimes\mathrm{End}(N) \longrightarrow \mathrm{End}(M\otimes N)$$

i.e. $u\cdot(m\otimes n) = \Delta(u)\cdot m\otimes n = \sum_i u_1^i\cdot m \otimes u_2^i \cdot n$.

M $\mathcal{U}$-module, then $M^* = \mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C})$ has $\mathcal{U}$-module structure:

$$(u\cdot f)(m) = f(S(u)\cdot m)$$

So we saw: Hopf algebra $\mathcal{U}$ $\rightsquigarrow$ monoidal structure on $\mathrm{Rep}\,\mathcal{U}$.

Now we want: quasi-triangular Hopf algebra $\mathcal{U}$ $\rightsquigarrow$ <u>braided</u> monoidal

E.g $\mathcal{U} = \mathcal{U}\mathfrak{g}$ cocommutative, then $\exists$ $\mathcal{U}$-equivariant isomorphism:

$$\tau : M \otimes N \to N \otimes M$$
$$m \otimes n \mapsto n \otimes m$$

This is coherent in the sense that:

$$\begin{tikzcd}
& M\otimes(N\otimes P) \arrow[r, "\tau"] & (N\otimes P)\otimes M \arrow[dr, "ass"] & \\
(M\otimes N)\otimes P \arrow[ur, "ass"] \arrow[dr, "\tau\otimes 1"'] & & & N\otimes(P\otimes M) \\
& (N\otimes M)\otimes P \arrow[r, "ass"'] & N\otimes(M\otimes P) \arrow[ur, "1\otimes\tau"'] &
\end{tikzcd}$$

"hexagon axiom".

If $\mathcal{U}$ non-cocommutative, need an ~~replacement~~ equivariant replacement for τ.

Def A Hopf algebra $\mathcal{U}$ is quasi-triangular if $\exists$ $\Theta \in \mathcal{U}\otimes\mathcal{U}$ invertible such that, $\forall$ $u \in \mathcal{U}$:

$$\Theta^{-1} \circ \Delta(u) \circ \Theta = (\tau \circ \Delta)(u). \qquad (1)$$

Prop Quantum groups $\mathcal{U}$ are quasi-triangular.

E.g $\mathcal{U}_q sl_2(\mathbb{C})$: $\quad \Theta = \sum_{n \geq 0} a_n F^n \otimes E^n \exp\left(\frac{h}{4} H \otimes H\right)$,

~~Actually need~~ $\qquad$ where $q = \exp(-\frac{h}{2})$, $K = \exp\left(-\frac{hH}{2}\right)$.

Consequences

1. $\Theta \circ \tau : M \otimes N \to N \otimes M$ is a $\mathcal{U}$-module isomorphism. Obviously bijective, then (1) implies, $\forall$ $u \in U$, $v \in M \otimes N$:

$$\Theta\circ\tau\,(u \cdot v) = \cancel{\Theta\circ(\tau\circ\Delta)(u)}\ \Theta\circ\tau\,(\Delta(u)\,v) = \Theta \circ (\tau\circ\Delta)(u)\ \tau(v)$$
$$\overset{(1)}{=} \Delta(u) \cdot \Theta\circ\tau(v) = u \cdot \Theta\circ\tau(v).$$

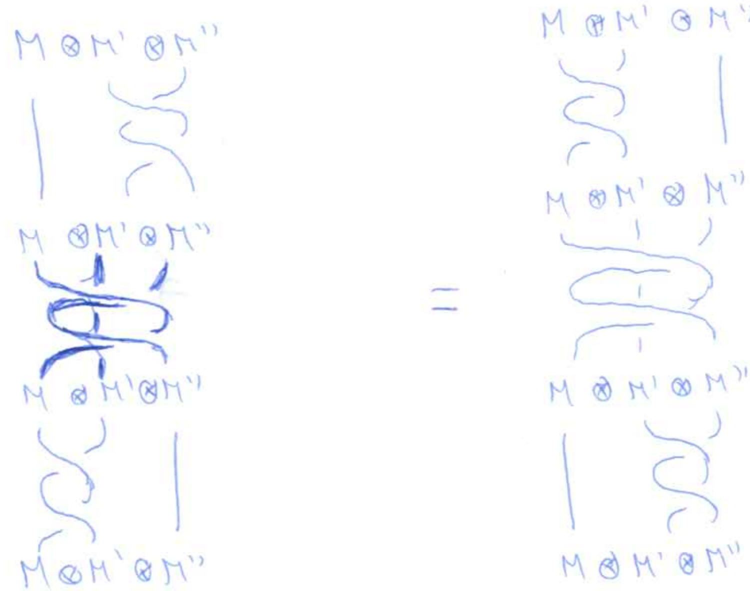
2. Θ is a solution to the quantum Yang-Baxter equation:

$$\Theta_{12} \circ \Theta_{13} \circ \Theta_{23} = \Theta_{23} \circ \Theta_{13} \circ \Theta_{12}$$

Here $\theta_{12}, \theta_{23}, \theta_{13}$ are endomorphisms of $M\otimes N \otimes P$:

$$\begin{tikzcd}
M\otimes M'\otimes M'' \arrow[r, bend left, "\theta_{12} = \theta\otimes 1"] \arrow[r, "\theta_{23} = 1\otimes\theta"] \arrow[r, bend right, "\theta_{13} = (1\otimes\tau)\circ(\theta\otimes 1)\circ(1\otimes\tau)"'] & M\otimes M'\otimes M''
\end{tikzcd}$$

QYB equation appears in statistical mechanics. Mathematically, it means we can think of multiplication by θ as some sort of braiding:



(Indeed, this gives a representation of the braid group B_3 on $M\otimes M'\otimes M''$.)

Note that QYB equation is highly overdetermined: if $M = M' = M''$ are f.d., then it's a system of $\dim(\mathrm{End}(M)^{\otimes 3}) = (\dim(M))^6$ equations in $\dim(\mathrm{End}(M)^{\otimes 2}) = (\dim(M))^4$ unknowns. So generally we don't expect solutions, but quantum groups provide them.

3. $\theta\circ\tau$ satisfies hexagon axioms.

2. Factorization homology

Target is a 2-category $\mathcal{C}$ (monoidal) of appropriate k-linear categories.

$$E_1\text{-}\mathrm{Alg}(\mathcal{C}) \cong \text{monoidal } k\text{-linear cat.}$$

$$E_2\text{-}\mathrm{Alg}(\mathcal{C}) \cong E_1\text{-}\mathrm{Alg}(E_1\text{-}\mathrm{Alg}(\mathcal{C})) \cong \text{braided monoidal } k\text{-linear cat.}$$

Standard argument, see e.g. Lurie DAG $\underline{\mathrm{VI}}$, Example 1.2.4.

Recall factorization homology (for surfaces): fix $\mathcal{A} \in E_2\text{-}\mathrm{Alg}(\mathcal{C})$, then:

$$(\mathbb{R}^2)^{\sqcup k} \longmapsto \mathcal{A}^{\boxtimes k}$$

$$\begin{tikzcd}
\mathrm{Disk}_2^{fr} \arrow[r] \arrow[d, hook] & \mathcal{C} \\
\mathrm{Mfd}_2^{fr} \arrow[ur, "\int \mathcal{A}"'] &
\end{tikzcd}$$

$\int \mathcal{A}$ = left Kan extension.

A few words about tensor product $\boxtimes$ on $\mathcal{C}$: given $\mathcal{A}, \mathcal{B} \in \mathcal{C}$, can define $\mathcal{A} \otimes \mathcal{B} \in \mathcal{C}$ with:

$$\mathrm{ob}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{B}),$$

$$\mathcal{A} \otimes \mathcal{B}\big((a,b),(a',b')\big) = \mathcal{A}(a,a') \otimes_k \mathcal{B}(b,b').$$

Want $\mathrm{Fun}(A \otimes B, C) \xrightarrow{\cong} \mathrm{Fun}(A, \mathrm{Fun}(B,C))$, but not true. Instead $\exists$ "completion" $A \boxtimes B$, with canonical functor $\boxtimes : A \otimes B \to A \boxtimes B$, which works in Rex = fin. co-complete k-linear cat, with right exact functors. This is Deligne-Kelly tensor product.

In section 3 of BZBJ, they claim it extends to Pr = comp. gen. presentable k-lin. cats, with compact & co-cts.

$$\mathrm{ind} : \mathrm{Rex} \overset{\sim}{\rightleftarrows} \mathrm{Pr} : \mathrm{comp}$$

equivalence of monoidal cats.

§5

<u>Def</u> The inclusion $\emptyset \to M$ induces a canonical functor:

$$\mathrm{Vect}_k = \int_{\emptyset} \mathcal{A} \to \int_M \mathcal{A}.$$

The image of the tensor unit $k \in \mathrm{Vect}_k$ under this functor is called the <u>distinguished object</u> or quantum <u>structure sheaf</u>, denoted $\mathcal{O}_{\mathcal{A},M}$.

<u>Prop</u> 1) $\mathcal{O}_{\mathcal{A},(\mathbb{R}^2)^{\amalg k}} \cong 1_{\mathcal{A}^{\boxtimes k}}$.

2) $M = X \coprod_{N \times \mathbb{R}} Y$ collar gluing, excision $\Longrightarrow \mathcal{O}_{\mathcal{A},M} \cong \mathcal{O}_{\mathcal{A},X} \boxtimes_{\int_N \mathcal{A}} \mathcal{O}_{\mathcal{A},Y}$.

Main object of study:

<u>Def</u> The <u>moduli algebra</u> of S (punctured surface) is

$$A_S := \underset{\sim}{\mathrm{End}_{\mathcal{A}}}(\mathcal{O}_{\mathcal{A},S}).$$

Main theorem:

<u>Thm</u> (5.11) $\int_S \mathcal{A} \simeq A_S\text{-}\underset{\sim}{\mathrm{mod}_{\mathcal{A}}}$; also gives combinatorial presentation of A_S.

Need to explain what the $\sim$ terms mean.

<u>3. Monads and module categories</u>

§3 <u>Def</u> ① A (right) $\mathcal{A}$-module category $\mathcal{M}$ for a tensor cat. $\mathcal{A}$ is $\mathcal{M} \in \mathrm{Pr}$, with action functor:

$$\mathrm{act} : \mathcal{M} \boxtimes \mathcal{A} \to \mathcal{M}$$
$$m \boxtimes X \mapsto m \otimes X. \qquad + \text{ assoc. axioms}$$

② For $m \in \mathcal{M}$, define:

$$\mathrm{act}_m : \mathcal{A} \to \mathcal{M}$$
$$a \mapsto m \otimes a.$$

This commutes with colimits, so it has right adjoint:

$$\mathrm{act}_m^R : \mathcal{M} \to \mathcal{A}.$$

③ $\underline{\mathrm{End}}_{\mathcal{A}}(m) := \mathrm{act}_m^R(m) = \mathrm{act}_m^R(\mathrm{act}_m(1_{\mathcal{A}}))$, <u>internal endomorphism algebra</u>.

④ $m \in \mathcal{M}$ is an <u>$\mathcal{A}$-progenerator</u> if act_m^R is faithful and preserves colimits.

(§4) <u>E.g.</u> Take $\mathcal{A}$ as an $\mathcal{A}^{\boxtimes n}$ module; then $1_{\mathcal{A}}$ is a progenerator.

$$\underline{\mathrm{End}}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}}) = \left(\bigoplus_{V \in \mathrm{comp}(\mathcal{A})} V^* \boxtimes V \right) \Big/ \langle \mathrm{Im}(\mathrm{id}_{W^*} \boxtimes \phi - \phi^* \boxtimes \mathrm{id}_V) \mid \phi : V \to W \rangle$$

(If $\mathcal{A} = \mathrm{Rep}\, \mathcal{U}g$, this is just direct sum over f.d. irreps:

$$\mathrm{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}}) = \bigoplus X^* \boxtimes X.\ )$$

The algebra structure is:

$$(V^* \boxtimes V) \otimes (W^* \boxtimes W) = (V^* \otimes W^*) \boxtimes (V \otimes W) \xrightarrow{R_{V^*,W^*} \boxtimes \mathrm{id}} (W^* \otimes V^*) \boxtimes (V \otimes W) \xrightarrow{\iota_{V \otimes W}} \underline{\mathrm{End}}_{\mathcal{A}}(1_{\mathcal{A}}).$$

<u>Def</u> $\mathcal{O}(\mathcal{A}) := T(\underline{\mathrm{End}}_{\mathcal{A}^{\boxtimes}}(1_{\mathcal{A}}))$, where $T : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ is the tensor functor. I.e. replace $\boxtimes$ with $\otimes$ everywhere.

<u>Thm</u> There is an equivalence of categories:

$$\mathcal{A} \underset{\mathcal{A} \boxtimes \mathcal{A}}{\boxtimes} \mathcal{A} \simeq \mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}.$$

<u>Sketch proof</u>

Uses Barr-Beck theorem for the adjoint pair:

$$\mathrm{act}_m : \mathcal{A} \rightleftarrows \mathcal{M} : \mathrm{act}_m^R$$

$S = \mathrm{act}_m^R \circ \mathrm{act}_m$ is ~~an~~ a monad (algebra-like functor):

$$\eta : \mathrm{id}_{\mathcal{A}} \to \mathrm{act}_m^R \circ \mathrm{act}_m \qquad \cancel{m : \mathrm{act}}$$

$$\mu : (\mathrm{act}_m^R \circ \mathrm{act}_m) \circ (\mathrm{act}_m^R \circ \mathrm{act}_m) = \mathrm{act}_m^R \circ (\mathrm{act}_m \circ \mathrm{act}_m^R) \circ \mathrm{act}_m \xrightarrow{\epsilon} \mathrm{act}_m^R \circ \mathrm{act}_m.$$

Get:

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathrm{act}_m"] \arrow[dr, "S"'] & \mathcal{M} \arrow[l, shift left, "\mathrm{act}_m^R"] \arrow[d, "\mathrm{act}_m^R", "\sim"'] \\ & S\text{-mod}_{\mathcal{A}} \end{tikzcd}$$

Barr-Beck gives necessary & sufficient conditions for $\widetilde{act}^R_m$ to be an equivalence:

- act^R_m is conservative, i.e. $act^R_m(X) \cong act^R_m(Y) \Longrightarrow X \cong Y$.
- act^R_m preserves certain colimits.

By definition, if m is a progenerator, then act^R_m is faithful and co-continuous, and Barr-Beck gives:

$$\mathcal{M} \simeq act^R_m \circ act_m\text{-mod}_{\mathcal{A}} \simeq \underbrace{act^R_m \circ act_m(1_{\mathcal{A}})}_{\underline{End}_{\mathcal{A}}(m)}\text{-mod}_{\mathcal{A}}.$$

One more leap: monadicity for base change: $F : \mathcal{A} \to \mathcal{B}$ (dominant) tensor functor, then:

$$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B} \simeq F(\underline{End}_{\mathcal{A}}(m))\text{-mod}_{\mathcal{B}}.$$

Apply to $T : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$, $\mathcal{M} = \mathcal{A}$, and progenerator $m = 1_{\mathcal{A}}$.

$$\Longrightarrow \mathcal{A} \underset{\mathcal{A}\boxtimes\mathcal{A}}{\boxtimes} \mathcal{A} \simeq T\left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)\text{-mod}_{\mathcal{A}} =: \mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}.$$

$\square$

§5

4. Gluing patterns

Want combinatorial model for surfaces as follows:

Def A *gluing pattern* is a bijection:

$$P : \{1, 1', \ldots, n, n'\} \to \{1, \ldots, 2n\},$$

such that $P(i) < P(i')$, $\forall\, i$.

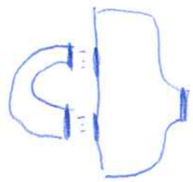
$P \rightsquigarrow$ surface $\Sigma(P)$, with marked boundary interval 0.

n handles

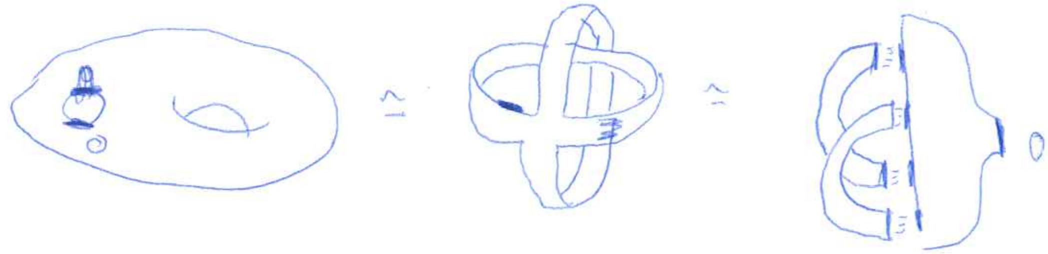
disk D^2, $2n+1$ boundary intervals

E.g: 1. Null gluing pattern $P: \emptyset \to \emptyset$, $\Sigma(P) \cong D^2 \cong S^2 - D^2$

2. $P: \{1,1'\} \to \{1,2\}$ $\quad \Sigma(P) \cong \mathrm{Ann}$



3. $P: \{1,1',2,2'\} \to \{1,3,2,4\}$ $\quad \Sigma(P) \cong T^2 - D^2$



4. P: $2g+r-1$ handles $\rightsquigarrow \Sigma(P) = \Sigma_{2g} - (D^2)^{\sqcup r}$

Get models for all orientable surfaces, punctured at least once.

Theorem (5.11) There is an equivalence of categories:

$$\int_{\Sigma(P)} \mathcal{A} \simeq a_P\text{-mod}_{\mathcal{A}},$$

where a_P is an algebra object in $\mathcal{A}$, isomorphic to $\mathcal{O}(\mathcal{A})^{\otimes n}$ (#handles) as objects, but with multiplication determined by P, in a way which will be specified later.

Sketch proof

- $\int_{D^2} \mathcal{A} \simeq \mathcal{A}$ as a category, but with $\mathcal{A}^{\boxtimes 2n}$-$\mathcal{A}$-bimodule structure, given by the markings. Denote it: $\int_{D^2} \mathcal{A} \cong {}_{\mathcal{A}^{\boxtimes 2n}}\mathcal{A}_{\mathcal{A}}$

- $\int_{H_1 \cup \ldots \cup H_n} \mathcal{A} \simeq \mathcal{A}^{\boxtimes n}_{\mathcal{A}^{\boxtimes 2n}, P}$ $\qquad (a_1 \boxtimes \ldots \boxtimes a_n) \boxtimes (b_1 \boxtimes \ldots \boxtimes b_{2n}) \mapsto$

$$(a_1 \otimes b_{P(1)} \otimes b_{P(1')}) \boxtimes \ldots \boxtimes (a_n \otimes b_{P(n)} \otimes b_{P(n')})$$

- excision $\Longrightarrow \int_{\Sigma(P)} \mathcal{A} \simeq \mathcal{A}^{\boxtimes n}_{\mathcal{A}^{\boxtimes 2n},P} \boxtimes_{\mathcal{A}^{\boxtimes 2n}} {}_{\mathcal{A}^{\boxtimes 2n}}\mathcal{A}_{\mathcal{A}}$.

- $\mathcal{A}^{\boxtimes n}_{\mathcal{A}^{\boxtimes 2n},P} \simeq \tau_P \cdot \left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)^{\boxtimes n}$ - mod$_{\mathcal{A}^{\boxtimes 2n}}$

- The action of $\mathcal{A}^{\boxtimes 2n}$ on ${}_{\mathcal{A}^{\boxtimes 2n}}\mathcal{A}_{\mathcal{A}}$ is given by iterated tensor product; using "monadicity for base change" (Thm. 4.11), get $\int_{\Sigma(P)} \mathcal{A} = T^{2n}\left(\tau_P \cdot \left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)^{\boxtimes n}\right)$ - mod$_{\mathcal{A}}$.

- Disregarding the algebra structure for a second:

$$T^{2n}\left(\tau_P \cdot \left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)^{\boxtimes n}\right) \underset{\text{as objects}}{\simeq} T\left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)^{\otimes n} \simeq \mathcal{O}(\mathcal{A})^{\otimes n}.$$

- But for multiplication, it's actually:

$$\cancel{\boxtimes_{i=1}^{n} T\left(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)}$$

$$\underbrace{\bigotimes_{i=1}^{n} \underbrace{T\left(\underline{End}_{\mathcal{A}_{P(i)} \boxtimes \mathcal{A}_{P(i')}}(1_{\mathcal{A}})\right)}_{=: \mathcal{O}(\mathcal{A})^{(i,i')}}}_{=: \mathcal{O}(\mathcal{A})^{(i)}}$$

~~Need the isomorphism btn. $\mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)}$ and $\mathcal{O}(\mathcal{A})^{(j)} \otimes \mathcal{O}(\mathcal{A})^{(i)}$~~

We need ~~to~~ to understand the isomorphism

$$c_{ij} : \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \to \mathcal{O}(\mathcal{A})^{(j)} \otimes \mathcal{O}(\mathcal{A})^{(i)},$$

because then multiplication is:

$$\begin{array}{c} \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \otimes \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \xrightarrow{1 \otimes c_{ij} \otimes 1} \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \\ \downarrow \text{mult} \otimes \text{mult} \\ \mathcal{O}(\mathcal{A})^{(i)} \otimes \mathcal{O}(\mathcal{A})^{(j)} \end{array}$$

C_{ij} could be the braiding of $\mathcal{A}$, or something more complicated. It's determined as follows. $\theta^{(i,i')}$ and $\theta^{(j,j')}$ commute in $\mathcal{A}^{\boxtimes 4}$, because they sit in different tensor factors.

E.g., for $P(1,1',2,2') = (1,3,2,4)$ "linked crossing", we have:

$$\theta^{(1,1')} \in \mathcal{A}^{\boxtimes 4} \quad \text{as} \quad \bigoplus_{V \in \mathrm{comp}(\mathcal{A})} V^* \boxtimes 1_{\mathcal{A}} \boxtimes V \boxtimes 1_{\mathcal{A}}$$

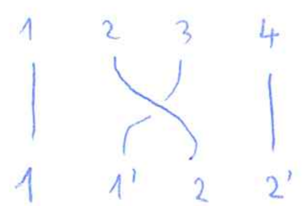
$$\theta^{(2,2')} \in \mathcal{A}^{\boxtimes 4} \quad \text{as} \quad \bigoplus_{W \in \mathrm{comp}(\mathcal{A})} 1_{\mathcal{A}} \boxtimes W^* \boxtimes 1_{\mathcal{A}} \boxtimes W$$

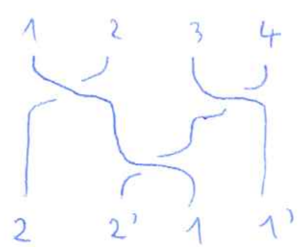
Regardless of ordering,

$$\theta^{(1,1')} \otimes \theta^{(2,2')} = \bigoplus_{V,W} V^* \boxtimes W^* \boxtimes V \boxtimes W = \theta^{(2,2')} \otimes \theta^{(1,1')}.$$

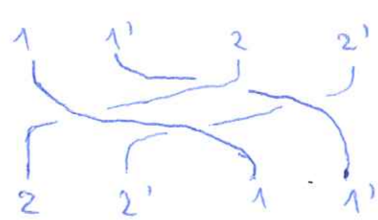
~~⇒ commutative di~~

$$\begin{tikzcd}
T^4\left(\theta^{(1,1')} \otimes \theta^{(2,2')}\right) \arrow[r, equal] & T^4\left(\theta^{(2,2')} \otimes \theta^{(1,1')}\right) \\
\theta(\mathcal{A})^{(1)} \otimes \theta(\mathcal{A})^{(2)} \arrow[u, "J_{12}"] & \theta(\mathcal{A})^{(2)} \otimes \theta(\mathcal{A})^{1} \arrow[u, "J_{21}"']
\end{tikzcd}$$

$J_{12} =$  $= 1 \otimes \sigma \otimes 1$ ← braiding of $\mathcal{A}$

$J_{21} =$  $= (\sigma \otimes \sigma) \circ (1 \otimes \sigma \otimes 1)$

Need $C_{12} = J_{12}^{-1} J_{21} = (1 \otimes \sigma^{-1} \otimes 1) \circ (\sigma \otimes \sigma) \circ (1 \otimes \sigma \otimes 1)$



This was the linked + case, $P(1) < P(2) < P(1') < P(2')$

There are 5 other cases to consider; we skip that.

$\boxtimes$

Eg. 1. $\int_{Ann} \mathcal{A} \simeq \mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}$. When $\mathcal{A} = \mathrm{Rep}\, \mathfrak{g}$, get modules over $\mathcal{O}(G)$.

2. $\int_{T^2 - D^2} \mathcal{A} \simeq \mathcal{D}(\mathcal{A})\text{-mod}_{\mathcal{A}}$. $\mathcal{D}(\mathcal{A})$ is the "elliptic double" of [Brochier-Jordan 14]. In nice cases, e.g. $\mathrm{Rep}\, \mathcal{U}_q \mathfrak{g}$, $\mathcal{D}(\mathcal{A})$ is isomorphic to the "Heisenberg double", which acts as an algebra of differential operators. Quantizes $\mathcal{U}_q \mathfrak{g} \ltimes \mathcal{O}(G)$.

3. S is genus g, punctured $n > 0$ times, then:

$$\Rightarrow \int_S \mathcal{A} \simeq \mathcal{D}(\mathcal{A})^{\otimes g} \otimes \mathcal{O}(\mathcal{A})^{\otimes (n-1)}\text{-mod}_{\mathcal{A}}$$

this is just the braiding in $\mathcal{A}$.

§6

5. Character Stacks

S – surface of genus g, $n>0$ pts. removed

$\Rightarrow \pi_1(S) \cong \mathbb{Z}^{*(2g+n-1)}$

$\underline{Ch}_G(S) \cong G^{2g+n-1}/G$ ← diag. action by conjugation.

↑ stacky quotient

$QCoh(*/G) \simeq Rep\, G$

$QCoh(\underline{Ch}_G(S)) \simeq \mathcal{O}(G^{2g+n-1})\text{-mod}_{Rep\, G}$

$\simeq \mathcal{O}(G)^{\otimes(2g+n-1)}\text{-mod}_{Rep\, G}$

No braiding to specify: Rep G is symmetric monoidal.

Thm 5.11 $\Longrightarrow QCoh(\underline{Ch}_G(S)) \simeq \int_S Rep\, G.$

This motivates the definition of quantum character stacks:

$$QCoh(\underline{Ch}_{U_q\mathfrak{g}}(S)) := \int_S Rep\, U_q\mathfrak{g}.$$

Just see this as deformation of functor:

$$QCoh(\underline{Ch}_G(-)).$$

§1

6. Building a TQFT:

What higher categorical structure do braided monoidal categories fit in?

Morita 3-category $MonCat_3$:

- Obj = braided monoidal cats
- 1-Hom(X,Y) = bimodule cats
- 2-Hom(M,N) = functors of bimodule cats
- 3-Hom(F,G) = nat. trans.

For q = root of unity,

$\mathrm{Rep}\, \mathcal{U}_q \mathfrak{g}$ is an example of fusion category, and these are (precisely?) the fully dualizable objects in MonCat_3. [Douglas, Schommer-Pries, Snyder 13].

Morita 4-category MonCat_4:

- Obj = braided tensor cat.
- 1-Hom(X,Y) = algebra objects in bimodule cat.
- 2-Hom(A,B) = bimodule cat. for the algebras
- 3-Hom(M,N) = functors
- 4-Hom(η,μ) = nat. transf.

[Freed-Teleman 12]: Modular categories (fusion + ribbon + condition) are fully dualizable, and moreover the fully ext. TQFT they define is invertible.

$\mathrm{Rep}\, \mathcal{U}_q \mathfrak{g}$ is modular if q = root of unity

Otherwise, get just "3+1 D TQFT", i.e. has the structure of a 4D TQFT, except not defined on 4-mfds. ← EXPECTATION, SEE WALKER'S NOTES

How does this TQFT look like?

$$* \longmapsto \mathrm{Rep}\, \mathcal{U}_q\mathfrak{g} = \mathcal{A} = \int_{\mathbb{R}^2} \mathcal{A}.$$

$$\bigcirc \longmapsto \mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}} \cong \int_{S^1 \times \mathbb{R}} \mathcal{A}.$$

$$\text{2-mfd } S \longmapsto \left(A_{S^0}\text{-mod-}1_{\mathcal{A}}\right)_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}} \simeq \int_S \mathcal{A}.$$

$$\text{3-mfd } M \longmapsto$$

II.§5

7. Closed surfaces

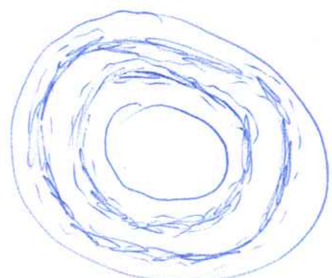
Given closed surface S, S° its puncture, we know $S = S^\circ \coprod_{Ann} D^2$.

$$\Rightarrow \int_S \mathcal{A} \simeq \int_{S^\circ} \mathcal{A} \boxtimes_{\int_{Ann}\mathcal{A}} \int_{D^2} \mathcal{A} \simeq A_{S^\circ}\text{-mod}_{\mathcal{A}} \boxtimes_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}} 1_{\mathcal{A}}\text{-mod}_{\mathcal{A}}.$$

Main technical result of the paper: we can compute these relative tensor products as

$$A_{S^\circ}\text{-mod}_{\mathcal{A}} \boxtimes_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}} 1_{\mathcal{A}}\text{-mod}_{\mathcal{A}} \simeq (A_{S^\circ}\text{-mod-}1_{\mathcal{A}})_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}}.$$

Idea: • $\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}$ is a braided tensor cat. (E_2 algebra in Rex or Pr), due to stacking product (§3):



• If $\mathcal{A}$ acts on module category $\mathcal{M}$, then $\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}$ acts:

§4 • $\mathcal{O}(\mathcal{A}) := T\left(\underline{\mathrm{End}}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})\right)$, and $1_{\mathcal{A}}$ maps to everything, ~~so~~ $\cancel{\mathcal{O}(\mathcal{A})}$ ~~maps to~~ in particular the pro-generator M of every module cat.

• $\mathcal{O}(\mathcal{A}) \xrightarrow{\mu} \underline{\mathrm{End}}_{\mathcal{A}}(M)$, quantum moment map. Induces $\cancel{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}} \to \mathcal{M}}$ $\simeq$ $\cancel{\underline{\mathrm{End}}_{\mathcal{A}}(M)\text{-mod}_{\mathcal{A}}}$

action of $\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}$ on $\mathcal{M} \simeq \underline{\mathrm{End}}_{\mathcal{A}}(M)\text{-mod}_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}}$ by relative tensor product:

$$\begin{aligned} \mathcal{M} \boxtimes \mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}} &\longrightarrow \mathcal{M} \\ M \boxtimes X &\longmapsto M \otimes_{\mathcal{O}(\mathcal{A})} X. \end{aligned}$$

• do the same with both left and right module cat, $\mathcal{M}$ and $\mathcal{N}$, get $\mathcal{M} \underset{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}}{\boxtimes} \mathcal{N} \simeq \left(\underline{\mathrm{End}}_{\mathcal{A}}(M)\text{-mod-}\underline{\mathrm{End}}_{\mathcal{A}}(N) \right)_{\mathcal{O}(\mathcal{A})\text{-mod}_{\mathcal{A}}}$.

Can also do this with marked pts. on S. Input data is $\mathcal{A}$-module cat. $\mathcal{M}_1, \dots, \mathcal{M}_k$ attached to pts $X = \{x_1, \dots, x_k\}$. Choosing a progenerator for each $\mathcal{M}_i$, i.e. $\mathcal{M}_i \simeq A_i\text{-mod}_{\mathcal{A}}$, get:

$$\cancel{\int_{(S,X)} (\mathcal{A}, \{\mathcal{M}_1, \dots, \mathcal{M}_k\}) \simeq (A_{S-X}\text{-mod} \dots)}$$

$$\int_{(S,X)} (\mathcal{A}, \{\mathcal{M}_1, \dots, \mathcal{M}_k\}) \simeq \int_{S-X} \mathcal{A} \underset{\int_{\coprod_k \mathrm{Ann}} \mathcal{A}}{\boxtimes} \int_{X \times D^2} \{\mathcal{M}_1, \dots, \mathcal{M}_k\}$$

$$\simeq \left(A_{S-X}\text{-mod-}(A_1 \boxtimes \dots \boxtimes A_k) \right)_{\mathcal{O}(\mathcal{A})^{\boxtimes k}\text{-mod}_{\mathcal{A}}}.$$