

# Invertible Topological Phases

1. Physics  $\rightsquigarrow$  Stable Homotopy Theory

Thm 1.1

$$\left\{ \begin{array}{l} \text{deform. classes of reflection positive} \\ \text{invertible } n\text{-dim extended TFTs,} \\ \text{w / symmetry } H_n \end{array} \right\} \cong [MTH, \Sigma^{n+1} \mathbb{Z}]_{\text{tor}}$$

spectra  
↙ ↘

Phase of matter := def. class of field theories

↓ (i), page 3  
low E approximation

↓  
long distance approx

↓  
scale invariant  $\rightsquigarrow$  conformal field theory

Gapped systems = nonzero energy levels have lower bound

↓ (ii), page 3

low energy approx. is topological

Phases of gapped syst.  $\cong$  def. classes of TFTs.

~~Symmetry~~ SPT phases  $\cong$  ~~invertible~~ def. classes of invertible TFTs

$$F(M^n) \neq 0$$

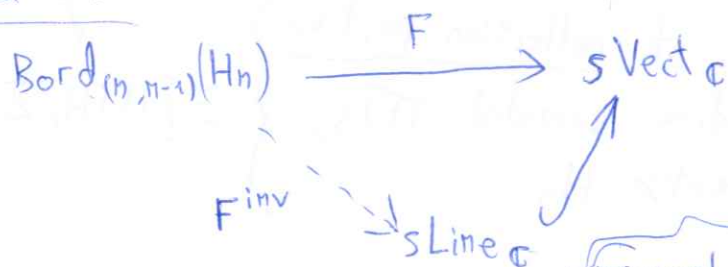
$$F(M^{n-1}) = V \cong \mathbb{C}$$

~~Tensor of TFTs  $F \otimes F'$  : superposition w/o interaction.~~

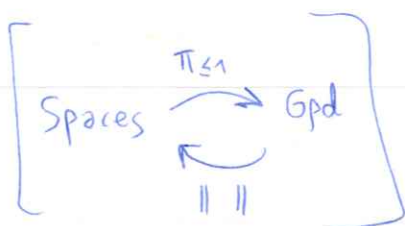
~~Tensor unit:  $1(M^n) = 1, 1(M^{n-1}) \cong \mathbb{C}$ ,~~

~~Def  $F$  is invertible if  $\exists F'$  st.  $F \otimes F' \cong 1$ .~~

Non-extended case:



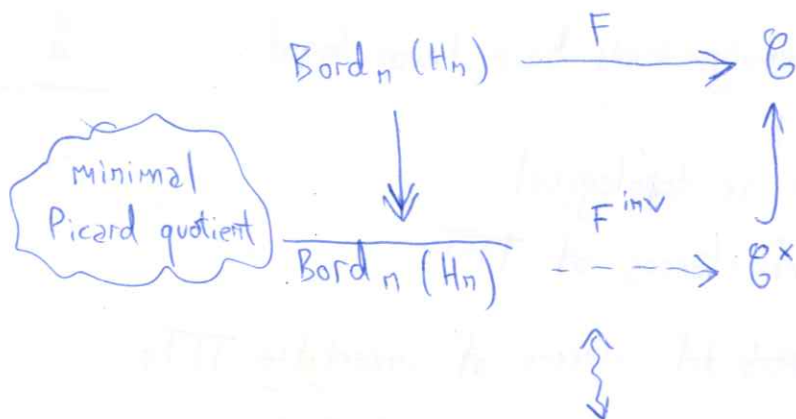
maximal Picard subgroup  
everything inv. under  $\otimes$



Let  $S = \|\text{sLine}_\mathbb{C}\|$ , then  $\pi_0 S \cong \mathbb{Z}/2\mathbb{Z}$ ,  $\pi_1 S \cong \mathbb{C}^\times$ ,  $\pi_i S = 0$ ,  $i > 1$

Now put usual topology on  $\mathbb{C}^\times$ , then  $\pi_0 S = \mathbb{Z}/2\mathbb{Z}$ ,  $\pi_1 S = 0$ ,  $\pi_2 S \cong \mathbb{Z}$ ,  $\pi_i S = 0$ ,  $i > 2$

Extended case:



Recall  $H_n$  structures:

$\|F\| : \|\text{Bord}_n(H_n)\| \rightarrow \|\mathcal{E}^\times\|$  map of spaces

But  $F$  symmetric monoidal btm. SMC  $\Rightarrow \|F\|$  map of  $E_\infty$ -spaces

May  $\Rightarrow$  can deloop spaces, get map of spectra.

## 2. Source & Target

Source

$$X := \|\overline{\text{Bord}_n(H_n)}\|$$

$\pi_0(X)$  = cob. classes of dim  $n$   $H_n$ -mfd's

$\Rightarrow X$  is the <sup>zero space of</sup> spectrum of  $H_n$ -cobordism.

$$(\sum_n MTH_n)_0$$

Construction:

(Pontrjagin-Thom)



$$\begin{array}{ccc} X_{n,n+q} & \rightarrow & B H_n \\ \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^{n+q}) & \rightarrow & B O_n \end{array}$$

$$(\sum_n MTH_n)_q = \text{Thom}(X_{n,n+q}; Q_q)$$

$Q_q$  pullback from quotient bundle over Grassmannian.

$$p \in \pi_0(\sum_n MTH_n) \rightsquigarrow S^{i+q} \xrightarrow{\phi} \text{Thom}(X_{n,n+q}; Q_q)$$

$\downarrow \cap$  0-section

$$M^i \subset S^{i+q}$$

$$TM^i \oplus \mathbb{R}^{n-i} \cong \phi^* V_n$$



$$[\phi] = 0 \iff M^i \cong \partial W^{i+1}$$

Target •  $\Sigma^n I\mathbb{C}^x$ , Pontrjagin dual to sphere spectrum.

$$[\mathbb{B}, \Sigma^n I\mathbb{C}^x] \cong \text{Hom}(\pi_n(\mathbb{B}), \mathbb{C}^x)$$

TFT det. up to iso. by  $\mathbb{Z}$

$$\pi_{\{0,1,\dots\}} S^0 \cong \{\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \dots\}$$

$$\pi_{\{0,-1,-2,\dots\}} I\mathbb{C}^x \cong \{\mathbb{C}^x, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \dots\}$$

• Anderson dual to  $S^0$ :

$$0 \rightarrow \text{Ext}^1(\pi_n B, \mathbb{Z}) \rightarrow [\mathcal{B}, \Sigma^{n+1} \mathbb{Z}] \rightarrow \text{Hom}(\pi_{n+1} B, \mathbb{Z}) \rightarrow 0$$

$$\pi_{\{0, -1, -2, \dots\}} \mathbb{Z} \cong \{ \mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, 0, \dots \}$$

$$\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^*$$

⊂ Eilenberg-MacLane

$$[\mathcal{B}, \Sigma^{n+1} \mathbb{Z}]_{\text{tor}} = \text{Ext}^1(\pi_n B, \mathbb{Z}) \leftarrow \text{Hom}(\pi_n B, \mathbb{C}^*)$$

E.g. Euler theory  
 $M^n \mapsto \lambda^{\text{Euler}(M)}$ ,  $\lambda \in \mathbb{C}^*$   
 $M^{n-1} \mapsto \mathbb{C}$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^* \rightarrow 0$$

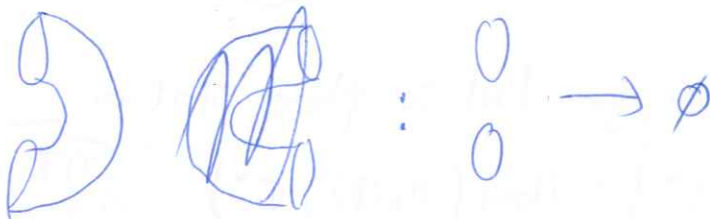
deform. classes ← iso. classes of inv. TFTs

Thm 5.20

$$\left\{ \begin{array}{l} \text{def. classes of inv.} \\ M\text{-dim TFTs, } H_n \end{array} \right\} \cong [\Sigma^n \text{MTH}_n, \Sigma^{n+1} \mathbb{Z}]_{\text{tor}}$$

### 3. Reflection positivity

Unitarity of Lorentzian FT  $\xrightarrow[\text{rotation}]{\text{Wick}}$  Reflection positivity of Euclidean FT



$$F \rightsquigarrow h: V \otimes V^* \rightarrow \mathbb{C}$$

Pos. def. hermitian metric

$$\Sigma^n \text{MTH}_n \rightarrow \Sigma^{n+1} \text{MTH}_{n+1} \rightarrow \dots$$

colimit MTH

Thm 8.23: Thm 1.1 again

E.g. Euler th. again  $\lambda \in \mathbb{R}^+$

$$\emptyset \xrightarrow{D^n} S^{n-1} \xrightarrow{D^n} \emptyset$$

$$h(F(D^n), F(D^n)) = \lambda^{\text{Euler}(S^n)} = \lambda^2$$

# 4. Classification

Symmetry:  $1 \rightarrow \left\{ \begin{matrix} \mathbb{Z}_2 \\ U(1) \\ SU(2) \end{matrix} \right\} \rightarrow H_n \rightarrow \left\{ \begin{matrix} O_n \\ SO_n \end{matrix} \right\} \rightarrow 1$  } time reversal  
} no time reversal

$U(1)$  "broken by particle-hole symmetry", e.g. in superconductors

$\Rightarrow$  only fermionic gps:  $\exists$  canonical morphism

$Spin_n \xrightarrow{f} H_n$ , fermionic iff  $f(-1) \neq 1$ .

$\Rightarrow$  10-fold way (9.2.1):

S	H	K
0	$Spin^c$	$\mathbb{T}$
1	$Pin^c$	$\mathbb{T}$

"complex"

S	H	K
0	$Spin$	$\mathbb{Z}/2$
-1	$Pin^+$	$\mathbb{Z}/2$
-2	$Pin^+ \times_{\mathbb{Z}_2} \mathbb{T}$	$\mathbb{T}$
-3	$Pin^- \times_{\mathbb{Z}_2} SU_2$	$SU_2$
4	$Spin \times_{\mathbb{Z}_2} SU_2$	$SU_2$
3	$Pin^+ \times_{\mathbb{Z}_2} SU_2$	$SU_2$
2	$Pin^- \times_{\mathbb{Z}_2} \mathbb{T}$	$\mathbb{T}$
1	$Pin^-$	$\mathbb{Z}/2$

"real"

$0 \rightarrow \mathbb{Z}/2 \rightarrow Spin^c \rightarrow SO_n \times U_1 \rightarrow 0$

## Free fermions

Free fermions  $\rightsquigarrow$  Clifford modules  $S$

$\exists$  admi  $m: S \otimes S \rightarrow \mathbb{R}$  mass

non-deg, skew-symm,  $H_n$ -invariant bilinear form

Thm 9.53  $\left\{ \begin{matrix} \text{free fermion theories} \\ \text{dim} = n-1, \text{ types, modulo} \\ \text{those with mass term} \end{matrix} \right\} \cong \pi_{3-s-n}(KO) \cong \pi_0(\Sigma^{n+s-3} KO)$   
 $\cong [KO, \Sigma^{n+s+1} \mathbb{Z}]$   
 $\cong [\Sigma^{-s} KO, \Sigma^{n+1} \mathbb{Z}]$

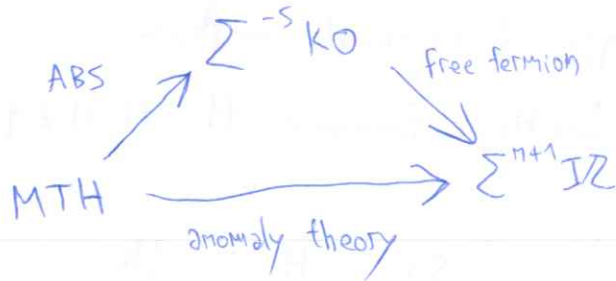
massless free fermion theories are anomalous;

U(1) charge conservation is incompatible w/ quantization

⇒ need to couple to bulk theory in  $n$ -dim

This can be taken topological invertible

Conjecture 9.62



Originally ABS: ~~Spin~~ MSpin → ~~Spin~~ KO

5. To do

$$\Sigma^n \text{MTH}_n \rightarrow \text{MTH} \rightarrow \Sigma^{n+1} \mathbb{Z}$$

$$\pi_i(\Sigma^n \text{MTH}_n) \rightarrow \pi_i(\Sigma^{n+1} \mathbb{Z})$$

choose lift to  $\Sigma^n \mathbb{I}\mathbb{C}^x$ ?  
 $F(S^n) = 1$ ?

$\left\{ \begin{array}{l} H_n\text{-cob. classes of} \\ \text{idim mfd} \end{array} \right\}$

?

Compute  $F(S^{n-1})$ ,  $F(S^i)$ ,  $\forall i$  : operators