# Invertible Topological Phases

## 1. Physics $\rightsquigarrow$ Stable Homotopy Theory

**Thm 1.1**

$$\left\{\begin{matrix}\text{deform. classes of } \underline{\text{reflection positive}} \\ \underline{\text{invertible}}\ n\text{-dim extended TFTs,} \\ \text{w/ symmetry } H_n\end{matrix}\right\} \cong \overbrace{[MTH, \Sigma^{n+1} I\mathbb{Z}]}^{\text{spectra}}_{\text{tor}}$$

Phase of matter $:=$ def. class of field theories

$\Big\downarrow$ (i), page 3

low E approximation

$\Big\downarrow$

long distance approx

$\Big\downarrow$

scale invariant $\rightsquigarrow$ conformal field theory

Gapped systems $=$ nonzero energy levels have lower bound

$\Big\downarrow$ (ii), page 3

low energy approx. is topological

Phases of gapped syst. $\cong$ def. classes of TFTs.

~~Symmetry~~ SPT phases $\cong$ ~~invertib~~ def. classes of invertible TFTs

$F(M^n) \neq 0$

$F(M^{n-1}) = V \cong \mathbb{C}$

~~Tensor of TFTs $F \otimes F'$ : superposition w/o interaction.~~

~~Tensor unit: $\mathbb{1}(M^n) = 1$, $\mathbb{1}(M^{n-1}) = \mathbb{C}$,~~

~~Def $F$ is invertible if $\exists\, F'$ s.t. $F \otimes F' \cong \mathbb{1}$.~~

Non-extended case:

$$\begin{tikzcd} \mathrm{Bord}_{(n,n-1)}(H_n) \arrow[r, "F"] \arrow[dr, dashed, "F^{inv}"'] & s\mathrm{Vect}_{\mathbb{C}} \\ & s\mathrm{Line}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

maximal Picard subgpd
everything inv. under $\otimes$

$$\left[\ \begin{tikzcd} \mathrm{Spaces} \arrow[r, bend left, "\pi_{\le 1}"] & \mathrm{Gpd} \arrow[l, bend left, "\|\ \|"] \end{tikzcd}\ \right]$$

Let $S = \| s\mathrm{Line}_{\mathbb{C}} \|$, then $\pi_0 S \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_1 S \cong \mathbb{C}^\times$, $\pi_i S = 0$, $i > 1$

Now put usual topology on $\mathbb{C}^\times$, then $\pi_0 S = \mathbb{Z}/2\mathbb{Z}$, $\pi_1 S = 0$, $\pi_2 S \cong \mathbb{Z}$, $\pi_i S = 0$, $i > 2$.

Extended case:

Recall $H_n$ structures:

$$\begin{tikzcd} \mathrm{Bord}_n(H_n) \arrow[r, "F"] \arrow[d] & \mathcal{C} \\ \overline{\mathrm{Bord}_n(H_n)} \arrow[r, dashed, "F^{inv}"] & \mathcal{C}^\times \arrow[u] \end{tikzcd}$$

minimal Picard quotient

$\Updownarrow$

$\|F\| : \| \overline{\mathrm{Bord}_n(H_n)} \| \longrightarrow \| \mathcal{C}^\times \|$ map of spaces

But $F$ symmetric monoidal ftr. SMC $\Longrightarrow$ $\|F\|$ map of $E_\infty$-spaces

$\xRightarrow{\text{May}}$ can deloop spaces, get map of spectra.

## 2. Source & Target

Source $X := \| \overline{\mathrm{Bord}_n(H_n)} \|$

$\pi_i(X)$ = cob. classes of dim $i$ $H_n$-mfds

$\Longrightarrow$ $X$ is the zero space of spectrum of $H_n$-cobordism.

$\overset{\shortparallel}{(\Sigma_n MTH_n)_0}$

Construction:
(Pontrjagin-Thom)

~~univ. bundle~~

$$\begin{array}{ccc} X_{n,n+q} & \dashrightarrow & BH_n \\ \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^{n+q}) & \rightarrow & BO_n \end{array}$$

$$(\Sigma_n MTH_n)_q = \mathrm{Thom}(X_{n,n+q}; Q_q)$$

$Q_q$ pullback from quotient bundle over Grassmannian.

$$\phi \in \pi_i(\Sigma_n MTH_n) \rightsquigarrow S^{i+q} \xrightarrow{\phi} \mathrm{Thom}(X_{n,n+q}; Q_q)$$

$\Big\downarrow \cap$ 0-section

$$M^i \subset S^{i+q}$$

$$TM^i \oplus \underline{\mathbb{R}}^{n-i} \cong \phi^* V_n,$$

$\begin{array}{c} V_n \\ \downarrow \\ X_{n,n+q} \end{array}$ tautological bundle

$$[\phi] = 0 \iff M^i = \partial W^{i+1}$$

Target • $\Sigma^n I\mathbb{C}^\times$, Pontrjagin dual to sphere spectrum.

$$[B, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}(\pi_n(B), \mathbb{C}^\times)$$

TFT det. up to iso. by $Z$

$$\pi_{\{0,1,\dots\}} S^0 \cong \{\mathbb{Z}, \mathbb{Z}/_2, \mathbb{Z}/_2, \mathbb{Z}/_{24}, 0, 0, \dots\}$$

$$\pi_{\{0,-1,-2,\dots\}} I\mathbb{C}^\times \cong \{\mathbb{C}^\times, \mathbb{Z}/_2, \mathbb{Z}/_2, \mathbb{Z}/_{24}, 0, 0, \dots\}$$

- Anderson dual to $S^0$:

$$0 \to \mathrm{Ext}^1(\pi_n B, \mathbb{Z}) \to [B, \Sigma^{n+1} I\mathbb{Z}] \to \mathrm{Hom}(\pi_{n+1} B, \mathbb{Z}) \to 0$$

$$\pi_{\{0,-1,-2,\dots\}} I\mathbb{Z} \cong \{\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \dots\}$$

$$I\mathbb{Z} \to I\mathbb{C} \to I\mathbb{C}^\times$$

Eilenberg-Maclane

$$[B, \Sigma^{n+1} I\mathbb{Z}]_{tor} = \mathrm{Ext}^1(\pi_n B, \mathbb{Z}) \leftarrow \mathrm{Hom}(\pi_n B, \mathbb{C}^\times) \cong [B, \Sigma^n I\mathbb{C}^\times]$$

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$$

E.g. Euler theory
$M^n \mapsto \lambda^{\mathrm{Euler}(M)}$, $\lambda \in \mathbb{C}^\times$
$M^{n-1} \mapsto \mathbb{C}$

deform. classes $\longleftarrow$ iso. classes of inv. TFTs

**Thm 5.20**

$$\left\{\begin{matrix}\text{def. classes of inv.} \\ n\text{-dim TFTs}, H_n\end{matrix}\right\} \cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}]_{tor}$$

## 3. Reflection positivity

Unitarity of Lorentzian ~~Q~~FT $\xrightarrow[\text{rotation}]{\text{Wick}}$ Reflection positivity of Euclidean FT

$\to \emptyset \quad \overset{F}{\rightsquigarrow} \quad h: V \otimes \overline{V} \to \mathbb{C}$

Pos. def. hermitian metric

$$\Sigma^n MTH_n \to \Sigma^{n+1} MTH_{n+1} \to \dots \qquad \text{colimit } MTH$$

**Thm 8.23**: Thm 1.1 again

E.g. Euler th. again $\lambda \in \mathbb{R}^\times$
$\emptyset \xrightarrow{D^n} S^{n-1} \xrightarrow{\overline{D^n}} \emptyset$
$h(F(D^n), \overline{F(D^n)}) = \lambda^{\mathrm{Euler}(S^n)} = \lambda^2$

# 4. Classification

Symmetry: $1 \to \begin{Bmatrix} \mathbb{Z}/2 \\ U(1) \\ SU(2) \end{Bmatrix} \to H_n \to \begin{Bmatrix} O_n \\ SO_n \end{Bmatrix} \to 1$ $\begin{Bmatrix} \text{time reversal} \\ \text{no time reversal} \end{Bmatrix}$

U(1) "broken by particle-hole symmetry", e.g. in supercomductors

only fermionic gps: ∃ canonical morphism

$\mathrm{Spin}_n \xrightarrow{f} H_n$, fermionic iff $f(-1) \neq 1$.

$\Longrightarrow$ 10-fold way (9.2.1):

| S | H | K |
|---|---|---|
| 0 | $\mathrm{Spin}^c$ | $\mathbb{T}$ |
| 1 | $\mathrm{Pin}^c$ | $\mathbb{T}$ |

"complex"

| S | H | K |
|---|---|---|
| 0 | Spin | $\mathbb{Z}/2$ |
| -1 | $\mathrm{Pin}^+$ | $\mathbb{Z}/2$ |
| -2 | $\mathrm{Pin}^+ \ltimes_{\{\pm1\}} \mathbb{T}$ | $\mathbb{T}$ |
| -3 | $\mathrm{Pin}^- \times_{\{\pm1\}} SU_2$ | $SU_2$ |
| 4 | $\mathrm{Spin} \times_{\{\pm1\}} SU_2$ | $SU_2$ |
| 3 | $\mathrm{Pin}^+ \times_{\{\pm1\}} SU_2$ | $SU_2$ |
| 2 | $\mathrm{Pin}^- \ltimes_{\{\pm1\}} \mathbb{T}$ | $\mathbb{T}$ |
| 1 | $\mathrm{Pin}^-$ | $\mathbb{Z}/2$ |

"real"

$0 \to \mathbb{Z}/2 \to \mathrm{Spin}^c \to SO_n \times U_1 \to 0$

## free fermions

Free fermions $\rightsquigarrow$ Clifford modules $S$

$\rightsquigarrow$ $m : S \otimes S \to \mathbb{R}$ mass

non-deg, skew-symm, $H_n$-invariant bilinear form

Thm 9.53 $\begin{Bmatrix} \text{free fermion theories} \\ \dim = n-1, \text{types, modulo} \\ \text{those with mass term} \end{Bmatrix} \cong \pi_{3-s-n}(KO) \cong \pi_0(\Sigma^{n+s-3} KO)$

$\cong [KO, \Sigma^{n+s+1} I\mathbb{Z}]$

$\cong [\Sigma^{-s} KO, \Sigma^{n+1} I\mathbb{Z}]$

massless free fermion theories are anomalous;
U(1) charge conservation is incompatible w/ quantization
$\Rightarrow$ need to couple to bulk theory in $n$-dim
This can be taken topological invertible

## Conjecture 9.62

$$\begin{tikzcd}
 & \Sigma^{-s} KO \arrow[dr, "\text{free fermion}"] & \\
MTH \arrow[ur, "ABS"] \arrow[rr, "\text{anomaly theory}"'] & & \Sigma^{n+1} I\mathbb{Z}
\end{tikzcd}$$

Originally $ABS : MSpin \longrightarrow KO$

## 5. To do

$$\Sigma^n MTH_n \longrightarrow MTH \longrightarrow \Sigma^{n+1} I\mathbb{Z}$$

$$\begin{tikzcd}
\pi_i(\Sigma^n MTH_n) \arrow[r] \arrow[d, leftrightarrow] & \pi_i(\Sigma^{n+1} I\mathbb{Z}) \arrow[d] \\
\{H_n\text{-cob. classes of } i\text{-dim mfds}\} & ?
\end{tikzcd}$$

choose lift to $\Sigma^n I\mathbb{C}^\times$?
$F(S^n) = 1$?

Compute $\cancel{F(S^{n-1})}$ $F(S^i)$, $\forall i$ : operators