Talk

- Preview: Want to construct Atiyah-type TFT with target category from derived (symplectic) algebraic geometry. origin

- Preliminaries: PTVV. n-shifted symplectic str.
- Short proof of Main thm.
- Application / Conjecture.

① Y : dArst. n-shifted Symplectic.

$$\begin{array}{ccc} \mathrm{Cob}_d^{or.(fr.)} & \longrightarrow & \mathcal{C} = \mathrm{Lagcorr}. \\ Z & \longmapsto & \mathrm{Map}((Z)_B, Y) \end{array}$$

: d-Art stack w/ $n-d+1$ shifted symplectic str.

$$X \quad (Z_1 \to X \leftarrow Z_2) \longmapsto \mathrm{Map}(X_B, Y) \longrightarrow \mathrm{Map}(Z_{1B}, Y) \times \mathrm{Map}(Z_{2B}, Y).$$

Induction map.

which is Lagrangian correspondence.

- My talk is mostly focused on construction of this functor, Mostly ⊛ Lagrangian correspondence !
- Main - Application: $Y = BG$, or similar type by Moore, Tachikawa

② Preliminaries

- Betti - stack

X: top'l space $\rightsquigarrow$ $X_B(S) = C^*_{sing}(X : k)$

$\Gamma(X_B, \mathcal{O}_{X_B}) = C^*_{sing}(X, k)$.

$\mathrm{Qcoh}(X_B) \sim (\infty, 1)$ Cat of locally const. sheaves of k-mod. on X

$\Gamma(X_B, \mathcal{F}) = \Gamma(X, \mathcal{F}) \quad \forall\, \mathcal{F} \in \mathrm{Qcoh}(X)$

. X is compact $\Rightarrow$ X_B is $\mathcal{O}$-compact.

(i.e. $\mathcal{O}_X \in D_{qcoh}(X)$ is compact obj.

perfect condition of $C(X, E)$. $\forall E$: per.

is every base change

Avoid unnecessary moiding

. X : $\mathcal{O}$-compact derived stack and $d \in \mathbb{Z}$.

An $\mathcal{O}$-orientation of deg d on X. is

$$[X] : C(X, \mathcal{O}_X) \longrightarrow k[-d] \quad ; \text{ fundamental class}$$

s.t. $\forall A \in cdga_k^{\leq 0}$, E on X_A, the morphism

$$- \cap [X_A] : C(X_A, E) \longrightarrow C(X_A, E^\vee)^\vee[-d] \quad \text{is q-iso of dg-module}$$

(PD)

(ex) X^d: cpt top. space $\rightsquigarrow$ X_B has d-orientation.

(Déf) $f : L \longrightarrow X$ a morp. of d. sf. (X, ω^n)

- isotopic str on f : path bet $f^*\omega \sim 0$ in $\mathcal{A}^{2,cl}(L, n)$
- $\mathrm{Isot}(f, \omega) \supseteq \mathrm{Lag}(f, \omega)$.
- Lag str on f is "non-degenerate" iso. str on f

$h \in \mathrm{Isot}(f, \omega)$

$\rightsquigarrow$ homotopy bet 0 & $T_L \wedge T_L \longrightarrow f^*(T_X) \wedge f^*(T_X) \longrightarrow \mathcal{O}_L[n]$

$$T_L \wedge T_f \qquad T_f \longrightarrow T_L \longrightarrow f^*(T_X)$$

Already equips with hom. to 0.

$\therefore$ Path(loop) in $\mathrm{Map}(T_L \wedge T_f, \mathcal{O}_L[n])$

$\therefore$ $\pi_1(\mathrm{Map}_{Lqcoh(X)}(T_f \otimes T_L, \mathcal{O}_L(n)), 0)$

$\cong$ $\Theta_h : T_f \longrightarrow \mathbb{L}_L[n-1]$. q-iso

(ex) if $X = *_{(n)}$, Θ_h gives $(n-1)$ shifted symplectic str on L

$\therefore$ (i.e) $f : L \longrightarrow X$ is Lag. $\Longleftrightarrow$ L has $(n-1)$ shifted str.

(Thm)
$$\begin{tikzcd} & L_1 \arrow[d, "f_1"] \\ L_2 \arrow[r, "f_2"'] & (X,\omega) \end{tikzcd} \Rightarrow \exists\ \mathbb{L}ag(f_1,\omega) \times \mathbb{L}ag(f_2,\omega) \to \mathrm{Sym}(L_1 \overset{h}{\times} L_2, n-1)$$

(i.e. $L_1 \times_{(X,\omega)} L_2$ has $(n-1)$ symplectic str.).

$\forall\ L \to X$, $A^{p.cl}(X/L, \omega) \to A^{p.cl}(X) \to A^{p.cl}(L)$.

$$X/L = \begin{tikzcd} L \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X/L \end{tikzcd}$$. L Lag $\Rightarrow$ we can lift ω to $\tilde{\omega}$

$L \times_X L$: stack of path from L to L

$\sim$ stack of pointed loop X/L.

$\sim$ $\mathrm{Map}(S^1, X/L) \rightsquigarrow n-1$ str

(Def) $f : \Upsilon \to \Sigma$ mor. bet $\mathcal{O}$-compact stacks.

$[\Upsilon]$ equipped w/. fundamental class

- Boundary str. on f is path from $f_*[\Upsilon]$ to 0.

  In $\mathrm{Map}(\Gamma(\Sigma, \mathcal{O}_\Sigma), k[-d])$

- $\mathbb{B}d(f, [\Upsilon])$: space.

- Non-degenerate means ~~that~~ "relative PD-condition"

- A relative d-orientation on $f : \Upsilon \to \Sigma$ is

  d-ori + non-degenerate boundary str.

(ex) $\overset{\overset{\sim}{\shortparallel}}{\partial M} \hookrightarrow \overset{\overset{\times}{\shortparallel}}{M}$

cpt ori.

$[M] \in H_{d+1}(M, \partial M, k) \longrightarrow H_d(\partial M, k) \longrightarrow H_d(M, k)$

$[M] \longmapsto [\partial M] = [\Upsilon] \longrightarrow f_*[\Upsilon]$

$\therefore$ $[M]$ determines $f_*[\Upsilon] \sim 0$ in $H_d(M, k)$

relative PD guarantees that $f : \gamma \to \Sigma$ boundary str

(thm) $f : \gamma \longrightarrow \Sigma$ (Y, ω^n) $\quad$ γ, Σ orientations

$$\mathbb{B}\mathrm{nd}(f, [\gamma]) \longrightarrow \mathbb{I}\mathrm{sot}\left(\mathrm{rest}, \int_{[\gamma]} ev_\gamma^* \omega\right)$$

where $\mathrm{rest} : \mathrm{Map}(\Sigma, Y) \longrightarrow \mathrm{Map}(\gamma, Y)$

$$\int_{[\gamma]} ev_\gamma^* \omega \in A^{2, cl}(\mathrm{Map}(\gamma, Y), n-d).$$

In particular, non-deg boundary condition $\Rightarrow$ Lagrangians.

$$\text{path } f_*[\gamma] \sim 0 \text{ in } \mathrm{Map}(P(Y, \mathcal{O}_X), k[-d])$$

$$\Downarrow$$

$$\text{path } \mathrm{rest}^* \int_{[\gamma]} ev_\gamma^* \omega \sim 0.$$

(check) well-definedness, non-degeneracy.

$$\begin{tikzcd}
\gamma \times \mathrm{Map}(\Sigma, Y) \arrow[r, "id \times \mathrm{rest}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \gamma \times \mathrm{Map}(\gamma, Y) \arrow[d, "ev_\gamma"] \\
\Sigma \times \mathrm{Map}(\Sigma, Y) \arrow[r, "ev_\Sigma"'] & Y
\end{tikzcd}$$

$$\odot \quad \mathrm{rest}^* \int_{[\gamma]} ev_\gamma^* \omega = \int_{f_*[\gamma]} ev_\Sigma^* \omega \underset{\uparrow}{\sim} 0$$

$(\because)$ $f_*[\gamma] \sim 0$.

why is this important.

(thm). $f : \gamma \longrightarrow \Sigma$, $g : L \longrightarrow Y$. dAr.

$$\mathrm{Map}(f, g) := \mathrm{Map}(\gamma, L) \overset{h}{\underset{\mathrm{Map}(\gamma, Y)}{\times}} \mathrm{Map}(\Sigma, Y)$$

f carries rel-dor & Y carries n-sym. g. lag

$\Rightarrow$ $\mathrm{Map}(f, g)$ has $(n-d-1)$ symplectic str if $X = *_k \Rightarrow$ obj $\Rightarrow$ $k-1$ symplectic d st..

Conjecture: Any obj Y in $\mathrm{Lag}_{(\infty,n)}(*_{(k)})$ fully dualizable

(i.e.) $\mathrm{Bord}_n^{fr} \longrightarrow \mathrm{Lag}_{(\infty,n)}(*_k)$. given by $\mathrm{Map}((-)_B, Y)$

$$\mathrm{Map}((pt)_B, Y) = Y$$

Also it factors through oriented version.

? <u>No evidence!</u>

③ Applications.

· Moore - Tachikawa conjecturally constructed 2d TFT. whose target category is coming from. symplectic holomorphic mfd.

$$\mathrm{Cob}_2 \longrightarrow \mathcal{E}$$

obj : Lie gp$/\mathbb{C}$ (alg - gp).

$T^*X[n]$ $= \mathbb{R}\mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_X}(T_X[-n])$

mor : holo symplectic manifold. w/

$$X \longrightarrow (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^* \text{ moment map}$$

i.e. X w/ hamiltonian action

$$id = T^*G_{\mathbb{C}}$$

In our case. this setting can be interpreted as follows

$$\mathcal{E} \longrightarrow \mathrm{Lag}_1 \quad \underline{\mathbb{C}^* \text{ action}}$$

$$G \longmapsto [\mathfrak{g}^*/G] = T^*[1]BG. \quad 1\text{-symplectic}$$

$$\begin{array}{ccc} X & & [X/G_1 \times G_2] \\ \downarrow & & \downarrow \ \mathrm{Lag} \\ (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^* & & [\mathfrak{g}_1^*/G_1] \times [\mathfrak{g}_2^*/G_2] \end{array}$$

composition. fiber product.

(Rmk). $X \to \mathfrak{g}^*$ moment map $\Rightarrow [X/G] \to [\mathfrak{g}^*/G]$ Lag.

pull back $\Rightarrow \mu^{-1}(0)/G$.

Our example. Lagcorr

$$Z^d \longrightarrow \mathrm{Map}(Z_B, Y). \quad (n-d.)$$

$$\begin{tikzcd} & X & \\ Z_1 \arrow[ur] & & Z_2 \arrow[ul] \end{tikzcd}$$

$\Rightarrow$ $Z_1 \amalg \bar{Z}_2 \longrightarrow X$ $\longrightarrow$ $\mathrm{Map}(X_B, Y) \xrightarrow{\ \text{l.g}\ } \mathrm{Map}(Z_1 \amalg \bar{Z}_2, Y)$

has 0-(rel) orientation $\quad = \mathrm{Map}(Z_1, Y) \times \mathrm{Map}(\bar{Z}_2, Y)$

$\| \qquad\qquad \overline{\mathrm{Map}(Z_2, Y)}$

Y n-deg $\longrightarrow$.

① Lag. okay

② $\mathrm{Map}(X_B, Y)$ d-Ar. stack.

③ Composition.

$$L_1 \xrightarrow{f_1} \underset{\omega_X \ \omega_Y}{X \times Y}. \qquad L_2 \xrightarrow{f_2} \underset{\omega_Y \ \omega_Z}{Y \times Z}$$

$$\Rightarrow \mathrm{Lag}(f_1, \pi_X^* \omega_X - \pi_Y^* \omega_Y) \times \text{"} \longrightarrow \mathrm{Lag}(g, \pi_X^* \omega_X - \pi_Z^* \omega_Z)$$

$$g : L_{12} = L_1 \times_Y L_2 \longrightarrow X \times Z \quad \text{Also Lag}$$

Almost Done!

So far, I described Atiyah type TFT, which has dAG origin.

One natural things comes up to the following.

("try to get" fully extended" Version)

(ex) Given k-symplectic X. ($= *_{(k)}$)

We can define $\mathrm{Lag}_{(\infty,0)}(X)$ of maps $Y \longrightarrow X$ Lag

If we are able to construct $\mathrm{Lag}_{(\infty,n)}(X)$, obj $Y \underset{\text{Lag}}{\hookrightarrow} X$

$\Rightarrow$ $\mathrm{Lag}_{(\infty, n+1)}(X)$. obj $Y \hookrightarrow X$

$\mathrm{Mor}(Y_1, Y_2) := \mathrm{Lag}_{(\infty,n)}(Y_1 \times_X Y_2)$

◎ Other Interesting examples

· Notation. Fact.

· G: cpt Lie gp $\longrightarrow$ $BG = [*/G]$ carries a 2-symplectic str.

· G is any Lie gp $\longrightarrow$ $[\mathfrak{g}^*/G] = T^*[1](BG)$.

✓ · G: cpt Lie gp $\longrightarrow$ $[G/G] = \mathrm{Map}(S^1_B, BG)$. 1-symplectic

① Classical CS theory $Y = BG$: (truncated to dim 1, 2).

G: compact / reductive.

$\uparrow$ $Z_{BG}: \mathrm{Cob}_2^{or} \longrightarrow \mathrm{LagCorr}_2$

dunno much about this theory

$S^1 \longrightarrow \mathrm{Map}((S^1)_B, BG) = [G/G]$ 1-Symplectic

$S^1 \times I \longrightarrow Z_{BG}(S^1 \times I) = D(G)$ Drinfeld Double

$$\begin{tikzcd} & {[G/G]} \arrow[dl] \arrow[dr] & \\ {[G/G]} & & {[G/G]} \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ a & & a^{-1} \end{tikzcd}$$

Pair of pants $\longrightarrow$ $[G \times G/G]$

$$\begin{tikzcd} & {[G\times G/G]} \arrow[dl] \arrow[dr] & \\ {[G/G]} & & {[G/G] \times [G/G]} \end{tikzcd}$$

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② CS Theory w/ G finite gp.

"Symplectic str won't play significant role"

3-2-1 $\quad$ a-2 cat

$Z_{BG}: \mathrm{Cob} \longrightarrow \mathrm{Corr}$ $\quad$ 2-Cat of monoidal Category

$S^1 \longrightarrow \mathrm{Qcoh}(\mathrm{Map}(S^1_B, BG)) = [G/G]$

$\Sigma \longrightarrow \mathrm{Map}(\mathrm{Qcoh}[G/G]^k, \mathrm{Qcoh}[G/G]^l)$

3d $\longrightarrow$ Natural transformation functor?

extend this to ◎ $\underbrace{\mathrm{Qcoh}(BG)}_{= \mathrm{Rep}\, G}$. 3-2-1-0 if G finite

② DW – theory.

Vect$_k$

$$N^{n-1} \longrightarrow \mathrm{Bun}_G(N^{n-1}) \xrightarrow{\text{locally const. fn}}$$

$$\begin{array}{ccc} & M^n & \\ \swarrow & & \searrow \\ N_1^{n-1} & & N_2^{n-1} \end{array} \longrightarrow \begin{array}{ccc} & \mathrm{Bun}_G(M) & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G(N_1) & & \mathrm{Bun}_G(N_2) \end{array} \xrightarrow[\text{push-forward}]{\text{pull-back}}$$

More precisely, space of sections

$$Z(M^{n-1}) = \text{sp. of sections of } \mathcal{L}_M.$$
$$= H^0(\mathrm{Map}(M, BG), \mathcal{L}_M).$$

~~Quasi-Ham~~ $\quad M^{n-1} \times \mathrm{Map}(M, BG) \xrightarrow{ev} BG$

$$\downarrow p$$

$$\mathrm{Map}(M, BG)$$

$$\int_{[M]} ev^* ?$$

Quasi-Ham str ($\sigma^{+}\theta$) <u>why ?</u>

, Symplectic str on this matters. <u>why ?</u>

$\Rightarrow \quad Z_{BG}(M) \quad M$: g - surface.

$\quad$ ‖

composition of M - 1pics? , & 1pics?

$$\Rightarrow Z_{BG}(M) = Z_{BG}(M') \times_{[G/G]} [pt/G]$$

This can also be obtained by Quasi-Hamiltonian reduction.

of $Z_{BG}(M') \simeq [\underbrace{G \times G \times \cdots G}_{2g \text{ times}} / G]$ deformation retract of ~~a~~ to wedge of 2g-circles

$\Rightarrow$

ex. <u>Quasi-Hamiltonian</u> space is Lag in $(G/G) = \mathrm{Map}(S^1_B, \underline{BG})$

$\Rightarrow$ cf.