(Baez – Dolan)

- GOAL : • Introduce (extended) TQFT & COBORDISM HYPOTHESIS.
- • Related concepts, idea & Get motivation.

(Next) • Proof (Outline). Examples.

(Setting) "+M : Smooth. cpt. (ori) mfd (with boundary).

Cob(d) : obj : (d−1) cld - mfds

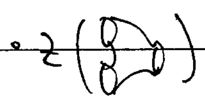
$\mathrm{Mor}(M,N) \cong \{B \mid \partial B \cong \bar{M} \amalg N\}/\text{diffeo}$

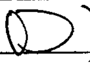
Composition ⟶ gluing. (Some issue is involved here) "Smooth str."

$\otimes$: disjoint union.

(Def) (Atiyah). A TQFT (of dim d) is a $\otimes$-functor $Z : \mathrm{Cob}(d) \to \mathrm{Vect}_{\mathbb{C}}$.

(ex) $(d=2)$. • $Z(S^1) = A$

• Z() $= m : A\otimes A \to A$. (A : com. asso)

• Z() $=$ unit $: \mathbb{C} \to A$. Z() $= tr : A \to \mathbb{C}$.

→ Trace pairing $A\otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$ non-deg.

In particular, A is f.d.

(thm) $\mathrm{Fun}^{\otimes}(\mathrm{Cob}(d), \mathrm{Vect}_{\mathbb{C}}) \cong \mathrm{Cat}(\text{FA})$

Also. $Z(S^2) \sim$ number.

$Z(g=1) \sim \dim A$.

① (diffeo) Inv

② Computational tool. ~ Not easy.

(∵) $d=2$ totally depend on. classification of manifold.

~~idea of build up. so that. get a nicer (manageable) tool.~~

For this. we need categorical formalism.

Explain. Cobordism Hypothesis.

(Write this) ⟶ Explain 2 Issues.

Two Issues

① Non-trivial tangent bundle.

(Def) M : m-dim'l mfd.

$\forall n \geq m$. define an n-framing of M to be a stabilized tangent bundle. $T_M \oplus \underline{\mathbb{R}}^{n-m}$

⇒ $Cob_n^{fr}(n)$ is defined the same way.

(CH). $\mathcal{C}$: s.m. d-Cat w/ tensor functor.

① $Z : Cob_d(d)^{fr} \longrightarrow \mathcal{C}$

② Ob in $\mathcal{C}$. with $Z(*) \in \mathcal{C}$.) ~~TFAE~~

finite condition = fully dualizable.

Higher Category. why — easier to define. / Useful in contemplate TQFT.

Want to use the language of (∞.n) Cat. Instead of n-Category

~~X top $\longrightarrow$ $\Pi_{\leq\infty}$ associative upto homotopy.~~

It has every good property we want to involve in. the definition of (∞.0) Category.

Define (∞.0) Cat = $\Pi_{\leq\infty} X$.

Inductively. say. (∞.1) Category = (∞.0) Cat enriched category.

Chain

Complexed-Valued. TQFT..

We don't actually care about this TQFT. however it is helpful to understand the idea.

→ Reason ①. $Z(M)$ given by hom or Cohomology

~~One try to think. axiomatize $Z(M)$~~ homology & Cohomology → $Z(M)$.

(Forget about extending down)

$Z(M) \longrightarrow$ Ch.

$Z(B) \longrightarrow$ Chain-map.

~~$B \simeq B'$. ⇒ $Z(B) = Z(B')$~~ Rather homotopy.

① Too strong

② In some cases. Chain-homotopy is not diff. invariant.

So. what we want is the following rules

cld $(n-1)$ mfd.	$\rightsquigarrow$	Chain-complex.
Bord	$\rightsquigarrow$	Chain-map.
diffeo.	$\rightsquigarrow$	Chain-homotopic.
Isotopy.		" of "
⋮		⋮

→ Extending up extension of TQFT.

$(\infty.1)$ Cat.

$$Z^{\otimes} : \widetilde{Cob}(n) \longrightarrow \mathcal{C}$$

To understand, need to precise def of $(\infty.1)$ Cat.

Before talking about this, Two ideas are combined together to give Bord$_n$ or Bord$_n^{fr}$ $(\infty.n)$ Category.

$(\infty.n)$ Category Bord$_n$.

ob	— pt		n-mor	: m with corners
1-Mor	— Bor.		n+1.	diff.
2-	Bor		n+2	: isotopic diff.

n-mor: classifying space of n- manifolds w/ corners.

(CH). $(\infty.n)$

$$Fun^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \cong \{\text{Fully dualizable obj}\}.$$

(Rmk) Non-trivial even if $n=1$. (See later)

(Optional) Complete Segal Space. throwing out non-inv. 1-mor.

Assume $(\infty,1)$ Cat $\mathcal{C}$. $\longrightarrow$ $(\infty,0)$ Cat X_0

called classifying space of objs.

Similarly $(\infty,1)$ $\mathrm{Fun}([1], \mathcal{C})$. $\longrightarrow$ $(\infty,0)$ Cat X_1

$\mathrm{Fun}^{\cong}([n], \mathcal{C})$ $\longrightarrow$ $(\infty,0)$ Cat X_n.

Claim: $\{X_n\}_{n\geq 0}$. remembers everything about $\mathcal{C}$.

- Simplicial space
- Str called Segal condition.

$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] \arrow[dr, phantom, "h.\circlearrowright"] & X_n \arrow[d] \\ X_m \arrow[r] & X_0 \end{tikzcd}$$

homotopy pull-back diagram.

i.e. $X_{m+n} \to X_n \times_{X_0} X_m$

$\to X_n \times^{h}_{X_0} X_m$

weak homotopy eq.

(Def) Segal space.

(prop) $\{X_n\}_{n\geq 0}$ Segal space. gives $(\infty,1)$ Category.

ex) $\mathrm{ob}\,\mathcal{C} \cong X_0$.

$x, y \in X_0$. $\mathrm{Map}_{\mathcal{C}}(x,y) = \{x\} \times^{R}_{X_0} X_1 \times^{R}_{X_0} \{y\}$.

(Rmk) Given $\{X_n\}_{n\geq 0}$. we get homotopy category X. (ordinary) hX.

obj: X_0

Mor. $\pi_0(\{x\} \times^{R}_{X_0} X_1 \times^{R}_{X_0} \{y\})$.

(Rmk2) Not one to one!

i.e. $\{Y_n\}_{n\geq 0} \longrightarrow \mathcal{C} \xrightarrow{\text{extract}} \{X_n\}_{n\geq 0}$.

$\exists$ a map $\underset{(\infty,0)}{Y_0} \longrightarrow \underset{(\infty,0)}{X_0}$.

1-morphism $\not\longleftrightarrow$ 1-morphism. each $N(\mathcal{C})_n$.

(ex) $\mathcal{C}$ ordinary Cat $\Rightarrow$ $N(\mathcal{C})$. endowed with discrete top.

is also segal space

By constr $\Rightarrow$ $\mathcal{C}_N \xrightarrow{(*)} X. \xrightarrow[\text{const}]{} \mathcal{C}_N$

~~However. $N(\mathcal{C}). \neq X.$~~ $\because$ Not discrete. $(\because)$

($\because$ ~~X_0. (conn comp) $\longleftrightarrow$ {iso class of obj}.~~

~~$N(\mathcal{C})$ no notion of isomorphic~~

(Def) $\varphi \in X_1$ inv $\Leftrightarrow$ $h\varphi$ (as 1-mor in $hX_\cdot$) is invertible.
$\{X_\cdot\}$ is complete. if the map $\delta: X_0 \to X_1$ induced by $\{0,1\} \to \{0\}$
is weak homotopy equivalence.
~~(i.e.) 1-morphism in X_0 is actually coming from Y_0.~~

(Def) An $(\infty,1)$ Cat is c.s.s. "

Fully dualizable obj.
(Observation 1) $\mathcal{C}$: monoidal.
$V \in \mathcal{C}$ has a right dual $V^\vee$ if
$\exists$ ev: $V \otimes V^\vee \to 1$ coev.: $1 \to V^\vee \otimes V$.
s.t. composition.
$$V \xrightarrow{id_V \otimes coev} V \otimes V^\vee \otimes V \xrightarrow{ev_V \otimes id} V = id_V$$
$$V^\vee \xrightarrow{coev_V \cdot id_{V^\vee}} V^\vee \otimes V \otimes V^\vee \xrightarrow{id_{V^\vee} \otimes ev} V^\vee = id_{V^\vee}$$

(Observation 2) $\mathcal{C}$: 2-Category.
Supp. $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ pair of functors are given.
Say. $u: id_{\mathcal{C}} \to G \circ F$. unit of adj. if
$\exists$ $v: F \circ G \to id_{\mathcal{D}}$. s.t.
(*) $\begin{cases} F \cong F \cdot id_{\mathcal{C}} \xrightarrow{id \cdot u} F \cdot G \cdot F \xrightarrow{v \cdot id} id_{\mathcal{D}} \cdot F \cong F. \\ G \cong id_{\mathcal{D}} \cdot G \longrightarrow \qquad \longrightarrow \end{cases}$

~~$\{$Adjunction$\} \longleftrightarrow \{u \cdot v$ with (*)$\}$.~~

So. for $B\mathcal{C}$ we can say
$(F \cdot G)$ adj $\Leftrightarrow$ $f \cdot g$ are dual) (ob3):
In general, $\mathcal{C}$ is (∞, n) Category. we say that
$f: X \rightleftarrows Y: g$ adj if $\exists$ $u \cdot v$ (*) holds up to iso

(Def) ① A (∞, n) Cat has adjoint if (sm) · $n=1$) Always
$n \geq 2$: every morphism
$n > 2$. $Hom_{\mathcal{C}}(x, y)$ $(\infty, n-1)$ has adjoints.
② $\mathcal{C}$ has dual if ① has adj.
② every obj. of $\mathcal{C}$ has a dual (in the sense that observation 1.
$\Leftrightarrow$ $B\mathcal{C}$ has adjoints.

(Def) Let $\mathcal{C}^{f.}$ be a largest subcat whose objs has duals.
if $X \in \mathcal{C}$ is fully dualizable obj if. it belong to the $\mathcal{C}^{full}$

(ex) $n=1$ X : fully dualizable if X is dualizable.
$n=2$. " if. X is. dualizable.
coev - ev have adjoints. //

(CH) $\mathcal{C}$ has a dual.
$Fun^{\otimes}(Bord_n^{fr}. \mathcal{C}) \cong \{.obj\} := \mathcal{C}^{\sim}$ (underly topoly space.

(Rmk) ① $Bord_n$., $Bord_n^{fr}$ has a dual. Change the orientation.
② eg Lat $(\infty.0)$ Category.

(Rmk) Non-trivial even if $n=1$.
$Z : Bord_1 \longrightarrow \mathcal{C}$.
cau. component
$Hom_{Bord_1}(\phi.\phi)$ = classifying space of. cld 1-manifold $\supseteq \mathbb{CP}^{\infty} = BU(1)$.
$Z \downarrow$
$Hom_{\mathcal{C}}(1.1) \longleftarrow$ encode what one-dim Z assigning to. the circle.
$BU(1)$. — $EU(1)$
classifying space for oriented circle-bundle.

As before, $Z(*) = X$ $Z(S^1) = \dim X$ not determine.
$Z|_{\mathbb{CP}^{\infty}}$. $S^1 \sim$ basic point of $\mathbb{CP}^{\infty}$. $\dim X$ base. point of $Hom(1,1)$
What we encodes is the circle action on. $\mathbb{CP}^{\infty}$ by. sym gp of circle.
Q: $\mathbb{CP}^{\infty}$ (BG) has G-action.

(ex) $\mathcal{C}$ $(\infty.1)$ Cat. obj : $\mathbb{C}$-alg
Mor : $Hom_{\mathcal{C}}(A.B) \cong$ chain-complex of A-B bimodule

(Rmk) Every obj. has fully dualizable $\to A^{op}$.
$Z(S^1) :=$ chain-complex $CH_*(A) \circlearrowleft S^1$.
unit obj = complex number $\Rightarrow$ $\mathbb{C}$-$\mathbb{C}$ bimodule. (Just chain complex - -)

∘ General version of M. (str)

(observation).

$O(n) \curvearrowright \{n\text{-framing on any mfd}\}$

$\Rightarrow \mathcal{E}O(n) \curvearrowright \mathrm{Bord}_n^{fr.}$

So, $\underset{\circlearrowleft O(n)}{\mathrm{Fun}^{\otimes n}(\mathrm{Bord}_n^{fr}, \mathcal{C})} \cong \{\text{fully}\}^{\circlearrowleft O(n)}$. underly ∞-gpoid (topological sp)

(ex) $n=1$. $\mathcal{C}$ has a dual.

$O(1) \cong \mathbb{Z}/2\mathbb{Z}$.

$\therefore \forall$ obj $X \in \mathcal{C}$ has a dual $X^\vee$

Take constr $X \longmapsto X^\vee$. ✓ gives $O(1)$-action.

ex) (G-structure)

① G : Arbitrary top'l space rep. $G \longrightarrow O(n)$

Define G-str on M

(i.e. P: G-bundle with Ass. v. bundle $P \times \mathbb{R}^n / G \cong T_M \oplus \mathbb{R}^{m-n}$

Similarly define Bord_n^G.

(ex) $\mathrm{Bord}_n^{fr} \cong \mathrm{Bord}_n^{\{1\}}$.

$\mathrm{Bord}_n^{or} \cong \mathrm{Bord}_n^{SO(n)}$.

$\mathrm{Bord}_n^{O(n)} \cong$ un-oriented mfd

~~$\Rightarrow$ Also, $G \curvearrowright \{\text{obj of } \mathcal{C}\}$ if $\mathcal{C}$ has a dual.~~

(CH for G).

$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \cong \{\text{obj in } \mathcal{C}\}^{hG}$.

$\cong \mathrm{Map}_G(EG, \{\text{obj}.\mathcal{C}\})$

(ex) $G = O(1)$.

$\mathrm{Fun}(\mathrm{Bord}_1, \mathcal{C}) \cong \{\text{obj}\}^{hO(1)}$.

$\cong$ Space of 'symmetrically "self dual" obj. in $\mathcal{C}$

X equipped with $X \otimes X \longrightarrow 1$ explicit X with dual as a of itself