MATH 312 (SUMMER II 2019): LINEAR ALGEBRA, ASSIGNMENT 1

Due: Monday 15th July, beginning of class.. If you want feedback on how you are going before deadline for dropping with no financial repercussions, you can submit up to five questions to me on Wednesday for feedback on Thursday. You can also come in to Thrusday office hours for feedback on questions – however I can not guarantee I will have sufficient time.

Note: Answers must be fully explained (unless otherwise specified), using full sentences for full credit. You are encouraged to work on these questions with your classmates, however please write down all people you discussed problems with on your assignment, and *write up all solutions on your own.* Asking these questions on online discussion boards is not acceptable.

Problems:

(1) Give all solutions to the following equations:

(a)

$$\begin{pmatrix} 3 & 3 & 3 \\ 4 & 2 & 2 \\ 1 & 4 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 18 \\ 18 \\ 12 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 3 & 2 & 12 \\ 1 & 1 & 5 \\ 0 & 2 & 6 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 3 & 2 & 12 \\ 1 & 1 & 5 \\ 0 & 2 & 6 \end{pmatrix} \vec{x} = \begin{pmatrix} 17 \\ 7 \\ 8 \end{pmatrix}$$

(2) Write down an equation for a circle in the plane passing through the points $(1, 7)$, $(6, 2)$, and $(4, 2)$. Is this circle unique?

(3) Compute the LU-decomposition of the matrix

$$\begin{pmatrix} 3 & 5 & 6 \\ 9 & 16 & 22 \\ 12 & 22 & 34 \end{pmatrix}.$$

Use this and back-substitution to solve the equation

$$\begin{pmatrix} 3 & 5 & 6 \\ 9 & 16 & 22 \\ 12 & 22 & 34 \end{pmatrix} \vec{x} = \begin{pmatrix} 23 \\ 78 \\ 114 \end{pmatrix}.$$

(4) Calculate the inverse of

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{3}{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} \end{pmatrix}$$

(5) Find (in terms of the parameter $\lambda \in \mathbb{R}$) all solutions to the equations

$$\begin{aligned} (1+\lambda)x_1 + x_2 + x_3 &= 1 \\ x_1 + (1+\lambda)x_2 + x_3 &= \lambda \\ x_1 + x_2 + (1+\lambda)x_3 &= \lambda^2 \end{aligned}$$

Note: If at any point you need to refer to the roots of some polynomial of degree 3 or higher you do *not* need to solve this polynomial. You may just refer to a root of such a polynomial in your answer.

(6) Suppose that for an equation $A\vec{x} = \vec{b}$, (with $\vec{x}, \vec{b} \in \mathbb{R}^3$, and A a 3 by 3 matrix), the full solution is given by

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

for $c \in \mathbb{R}$.

(a) Let $R\vec{x} = \vec{d}$ be the row reduced echelon form of the equation[1]. What is R, what is $\vec{d}$?

(b) If $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ have solutions, what is the column space of A?

(7) If possible write down a matrix with each of the following sets of properties. Explain why the matrix you write down has the property in question. If not possible explain why it is not possible.

(a) A 2×3 matrix with a row space of dimension 1 and a column space of dimension 2.

(b) A 3×3 matrix with a null space of dimension 1.

(c) A 3×3 matrix with a null space of dimension 1 and a column space of dimension 1.

(d) A 3×2 matrix A, such that any 2×2 minor (ie a matrix consisting of two of the rows of A) is invertible, such that the equation

$$A\vec{x} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

has a solution ($\vec{x} \in \mathbb{R}^2$). (Hint: Draw a diagram showing the three equations imposed by this matrix as lines in the plane).

(e) A 3×3 matrix such that the equation

$$A\vec{x} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

($x \in \mathbb{R}^3$) has a unique solution and the equation

$$A\vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

($y \in \mathbb{R}^3$) has infinitely many solutions.

(f) A 3×2 matrix A such that the null space has dimension 1, and the null space of A^T has dimension 1.

(8) Suppose that we have matrices A and B, and vectors $\vec{x}$, $\vec{z}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ such that

- $A\vec{x} = \vec{b}$.
- $B\vec{b} = \vec{c}$.
- $B\vec{x} = \vec{d}$.
- $A(\vec{x} + \vec{z}) = \vec{a}$.

For each of the following equations write down a solution (for $\vec{u}$, $\vec{v}$, $\vec{w}$) if if is possible with the above information. Otherwise state that it is not possible (an explanation of why is not necessary):

- $BA\vec{u} = 4\vec{c}$.
- $AB\vec{u} = \vec{c}$.

[1] Note that this hides an assumption that we did not swap any rows in getting from the original equation to the row reduced echelon form of it.

- $A\vec{v} = \vec{a} - \vec{b}$.
- $(A - B)\vec{w} = -\vec{b} + \vec{d}$.

(9) Describe the column space of a 3×3 matrix A, given that the equations $A\vec{x} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ and $A\vec{x} = \begin{pmatrix}0\\0\\2\end{pmatrix}$ each have infinitely many solutions.

(10) Write down a matrix equation $D\vec{x} = \vec{y}$ describing the point $\vec{x} = (x_0, x_1)$ such that translating by (ie adding the vector) $(3, 2)$, and then rotating by $\frac{\pi}{3}$ radians counterclockwise around the point $(2, 1)$ gives the point $(5, 6)$. What is the null space of D? What is the column space of D? What is the rank of D?

(11) Is there a 2×2 real matrix that transforms the vertices $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ to each of the following sets? If so write down such a matrix. If not, explain why not.

(a) $\{(0, 0), (1, 1), (-1, 1), (0, 2)\}$.
(b) $\{(1, 0), (1, 1), (2, 0), (2, 1)\}$.
(c) $\{(0, 0), (1, 1), (2, 1), (1, 0)\}$.
(d) $\{(0, 0), (3, 2), (4, 5), (7, 8)\}$.
(e) $\{(0, 0), (\sqrt{2}, 0), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$.

(12) Consider the (directed) graphs in figures 1 and 2: Questions (a), and (b) can be answered for *either* Graph 1 *or* Graph 2. Questions (c)-(e) must be answered for graph 2.

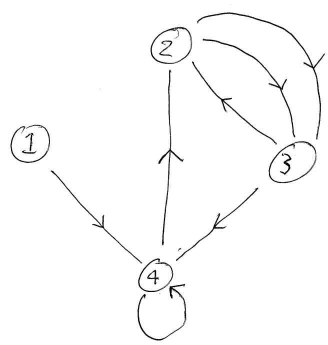

FIGURE 1. Graph 1

(a) What is the Adjacency matrix for either the graph in figure 1 or the graph in figure 2?
(b) For each pair of vertices how many paths of length 3 are there between them?
(c) What is the incidence matrix?
(d) What is the null space of the incidence matrix?
(e) What is the row space of the incidence matrix?
(f) (Harder) Show that if a (directed) graphs (for a graph with a finite number of vertices, and more than one vertice) underlying undirected graph is a tree (ie. after removing directional arrows on the edges the resultant graph contains no loops) then the Null space of the incidence matrix is $\vec{0}$.

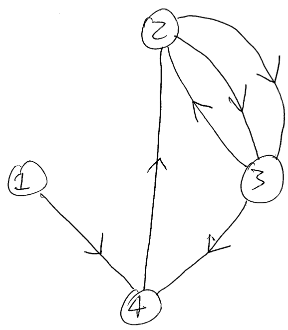

FIGURE 2. Graph 2

(13) Suppose you have a garden. This garden has trees that are one year old, eleven years old, and twenty-one years old. Every ten years you replace each dead tree with a one year old tree. Assume for simplicity that these are the only new trees added to the garden (that is to say no trees grow there naturally).
- A tree that is one year old has a 3/4 probability of surviving the next ten years.
- A tree that is eleven years old has a 1/2 probability of surviving the next ten years.
- A tree that is twenty-one years old has 0 probability of surviving the next ten years.

We can describe the change in the age of the tree population as a Markov process. Write down a matrix describing this Markov process.

What is a steady-state solution for this Markov process? Explain what this means in terms of the gardens tree population.

(14) Either given an example, or explain why no examples exist, of a Markov chain with multiple steady state solutions. If you give an example explain why your example has multiple steady state probability distributions.

Reminder: A Steady state probability distribution is a vector x, such that $Ax = x$, $\sum x_i = 1$ and all entries positive.

(15) Either give an example, or explain why no examples exist, of a Markov chain with at least one steady state probability distribution, and an $n \times n$ transition matrix A, such that the dimension of the column space of $A - Id$ is n.