MATH 312 (SUMMER II 2019): LINEAR ALGEBRA, ASSIGNMENT 2

Due: Monday 22nd July, beginning of class..

Note: Questions 9,11, and 12 have a significantly higher number of points attached to them.

Note: Answers must be fully explained (unless otherwise specified), using full sentences for full credit. You are encouraged to work on these questions with your classmates, however please write down all people you discussed problems with on your assignment, and *write up all solutions on your own.* Asking these questions on online discussion boards is not acceptable.

(1) (a) Consider the plane in $\mathbb{R}^3$, spanned by the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$. What is the minimum distance between a point on this plane and the point $\begin{pmatrix}6\\-2\\5\end{pmatrix}$?

(b) What is the angle between the vectors $\begin{pmatrix}1\\4\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\\3\end{pmatrix}$?

(c) Consider the line $(x, y, z)^T = (2t, 3t, -t)^T$ (recall e.g. $(2t, 3t, -t)^T = \begin{pmatrix}2t\\3t\\-t\end{pmatrix}$). What is a line (passing through the origin) perpendicular to this line and lying in the plane spanned by the vectors $(9, 4, -1)^T$ and $(7, 1, 0)^T$?

(2) Which of the following are vector spaces. Either show they are vector spaces, or show they are not.

(a) The set of functions[1] $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication of functions defined in the normal way, that is for $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$, and $x \in \mathbb{R}$:

$$(f + g)(x) = f(x) + g(x),$$

$$(af)(x) = a(f(x)).$$

(b) The set $V \times W$, where V and W are vector spaces over k. Addition is given by $(\vec{v_1}, \vec{w_1}) + (\vec{v_2}, \vec{w_2}) = (\vec{v_1}+\vec{v_2}, \vec{w_1}+\vec{w_2})$, and scalar multiplication (by $a \in k$) is given by $a(\vec{v}, \vec{w}) = (a\vec{v}, a\vec{w})$. Note: This vector space is denoted by $V \oplus W$.

(c) $\mathbb{R}^2$, with the normal addition, but with scalar multiplication given by

$$c\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}cx_1\\x_2\end{pmatrix}.$$

(d) The complex numbers over the real numbers (with the usual multiplication and addition). That is, is $\mathbb{C}$ a $\mathbb{R}$-vector space?

(e) $Hom(V, W)$ – The space of Linear functions between two vector spaces V and W (over k). Usual addition and multiplication of functions (as in (a)).

(f) Invertible 3×3 real matrices.

(g) The space of solutions to the differential equation $\frac{d^3y}{dx^3} = y$ on some interval (normal addition, scalar multiplication of functions).

[1] *Note:* We could also replace the word functions here with either "continuous functions" or "differentiable functions" without changing the answer, however the fact that e.g. the sum of two continuous function is continuous is really a question of analysis and outside the scope of this course.

(h) The space of solutions to $\frac{d^3y}{dx^2} = y^2$ on some interval (normal addition, scalar multiplication of functions).

(3) Consider the set of functions[2] $f : [0,1] \to \mathbb{C}$ as a vector space over the complex numbers. Equip it with the inner product
$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$
Show that the functions $\{e^{2\pi inx} | n \text{ an integer. ie. } n = ... - 2, -1, 0, 1, 2, ...\}$ are orthonormal (we call a set of functions $f_{\alpha \alpha \in S}$ orthonormal if $||f_\alpha|| = 1$ for all $\alpha \in S$, and $\langle f_\alpha, f_\beta \rangle = 0$ if $\alpha \neq \beta$).

(4) *Discrete Fourier Series.* Show that the set $\{\vec{v_1}, ..., \vec{v_n}\}$ of $\mathbb{C}^n$ given by
$$(\vec{v_k})_j = \frac{1}{\sqrt{n}} e^{2\pi ik(j-1)/n}$$
(that is the j^{th} entry of the vector $\vec{v_k}$ is $\frac{1}{\sqrt{n}} e^{2\pi ik(j-1)/n}$) is an orthonormal basis for $\mathbb{C}^n$ (with respect to the standard inner product on $\mathbb{C}^n$; $\langle (a_1, a_2, ..., a_n)^T, (b_1, ..., b_n)^T \rangle := \sum_{i=1}^n a_i \overline{b_i}$).

Hint: Use the formula for partial sums of geometric series.

Write the coordinate vector $\vec{e_1} = (1, 0, 0, .., 0)^T$ in this basis.

(5) Which of the following functions are inner products on the given vector space, of for (c)-(e) satisfy all the axioms of an inner product space except the condition; $\langle f, f \rangle = 0$ only if $f = 0$:

(a) On $\mathbb{R}^2$, $\langle x, y \rangle := x^T Ay$ for $A = \begin{pmatrix} 3 & 1 \\ 0 & -4 \end{pmatrix}$.

(b) On $\mathbb{R}^2$, $\langle x, y \rangle := x^T Ay$ for $A = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$.

(c) On the set of functions[3] $f : [0,1] \to \mathbb{C}$ as a vector space over the complex numbers, the function
$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}w(x)dx$$
where $w(x) = 1 + x^2$.

(d) On the set of functions[4] $f : [0,1] \to \mathbb{C}$ as a vector space over the complex numbers, the function
$$\langle f, g \rangle := \int_0^1 f(x)[\overline{g(x)}]^2 dx.$$

(e) On the set of functions[5] $f : [0,1] \to \mathbb{C}$ as a vector space over the complex numbers, the function
$$\langle f, g \rangle := \int_0^1 [f(x)\overline{g(x)}]^2 dx.$$

(6) Calculate the matrix[6] describing rotation (your choice of direction) of $\frac{\pi}{3}$ radians around the line[7]
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}$$
in $\mathbb{R}^3$.

(7) Consider graph 1 from question 12 of Assignment 1. How many paths of length 100 are there between any two vertices?

[2]Strictly speaking "L^2" -functions. This means that the necessary integrals are defined and converge, and solves the problem that this doesn't satisfy $\langle f, f \rangle = 0$ if and only if $f = 0$ if we work with the usual notion of function.

[3]Strictly speaking integrable functions.

[4]Strictly speaking integrable functions.

[5]Strictly speaking integrable functions.

[6]With respect to the standard basis of $\mathbb{R}^3$.

[7]That is restricting to each plane orthogonal to this line, we get a rotation of $\frac{\pi}{3}$ radians around the point of intersection with this line.

(8) Consider $\mathbb{C}^3$ with the standard inner product. Fix a matrix A. Show that there is a unique matrix A^* such that

$$\langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^*\vec{y}\rangle$$

for all $\vec{x}, \vec{y} \in \mathbb{C}^3$. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, what is A^*?

Note, we call A^* the *adjoint* of A.

(9) Let V be a finite dimensional vector space over a field k, and let $V^* := Hom(V, k)$, where by $Hom(V, k)$ we mean the set of linear functions $f : V \to k$, with addition and scalar multiplication defined in the usual way for functions. Given a basis $\{\vec{v}_1, ..., \vec{v}_n\}$ of V we can define elements $\delta_{\vec{v}_i} \in V^*$ as the unique element[8] of $V*$ such that

$$\delta_{\vec{v}_i}(\vec{v}_j) = \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{otherwise.} \end{cases} \tag{0.1}$$

(a) Explain why equation 0.1 uniquely defines an element of V^*. What is

$$\delta_{\vec{v}_i}(a_1\vec{v_1} + a_2\vec{v_2} + ... + a_n\vec{v_n})$$

(b) Show that $\{\delta_{\vec{v}_1}, ..., \delta_{\vec{v_n}}\}$ is a basis of V^*.

(c) Show that there is then a unique linear isomorphism[9] $\Phi : V \to V^*$, with $\Phi(v_i) = \delta_{v_i}$.

(d) Does the map Φ depend on the choice of basis?

(e) If V is a real[10] inner product space, show that the map $V \to V^*$, $\vec{v} \mapsto \langle -, \vec{v}\rangle$ is a linear transformation. Note that $\langle -, \vec{v}\rangle : V \to k$ is the function that maps $\vec{w} \mapsto \langle \vec{w}, \vec{v}\rangle$ for $\vec{w} \in V$.

(f) (Harder) Using the same procedure and the basis $\{\delta_{\vec{v}_1}, ..., \delta_{\vec{v}_n}\}$ of V^* we can define a linear isomorphism $\Psi : V^* \to (V^*)^*$. Show that $\Psi \circ \Phi : V \to (V^*)^*$ does *not* depend on the choice of basis we started with.

(g) Let V, W be two vector spaces with bases $\{\vec{v}_1, ..., \vec{v}_n\}$ and $\{\vec{w}_1, ..., \vec{w}_m\}$. Show that a basis of $Hom(V, W)$ is given by $\{\vec{w}_j\delta_{\vec{v}_i} | 1 \le i \le n, 1 \le j \le m\}$, where $\vec{w}_j\delta_{\vec{v}_i}(\vec{v}) = \delta_{\vec{v}_i}(\vec{v})\vec{w}$.

(h) Let $V = \mathbb{R}^2$, $W = \mathbb{R}^2$ with the standard bases. Write down the matrices corresponding to the basis vectors for $Hom(\mathbb{R}^2, \mathbb{R}^2)$ described in the previous part of this question.

(i) If we have a linear function $f : V \to W$, show that the map

$$f^* : W^* \to V^*$$

defined below is linear. Let $\delta \in W^*$, that is δ is a linear function $\delta : W \to k$. Let $\vec{v} \in V$. We define f^* by

$$(f^*(\delta))(\vec{v}) = \delta(f(\vec{v}))$$

or equivalently by

$$f^*(\delta) := \delta \circ f : V \to k.$$

(10) Consider the basis

$$\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \}$$

of $\mathbb{R}^3$. Produce an orthonormal basis from this using the Gram–Schmidt method. What is the base change matrix between the standard basis and this basis?

[8] Note that while $\delta_{\vec{v}_i}$ is a vector we are simplifying notation by not writing an arrow above it.

[9] A *linear isomorphism* is a linear map $\Phi : V \to W$, such that there exists an inverse $\Psi : W \to V$, such that $\Psi \circ \Phi = Id_V$, $\Phi \circ \Psi = Id_W$. Equivalently Φ is an isomorphism if $Im(\Phi) = W$ ($Im(\Phi)$ denotes the image of Φ, we have also sometimes called this the column space of Φ), and $Ker(\Phi) = \{\vec{0}\}$ ($Ker(\Phi)$ denotes the Kernel of Φ, we have also called this the null space of Φ).

[10] That is the underlying vector space is a vector space over $\mathbb{R}$.

(11) Quotient Vector Spaces: Let W be a *vector subspace* of V. Consider the relation [11] $\sim_W$ on V, defined by $\vec{v_1} \sim_W \vec{v_2}$ if and only if $\vec{v}_1 - \vec{v}_2 \in W$.

Show that:

(a) If $\vec{v}_1 \sim_W \vec{v}_2$ then $\vec{v}_2 \sim_W \vec{v}_1$ for all $\vec{v}_1, \vec{v}_2 \in V$.

(b) We have $\vec{v} \sim_W \vec{v}$ for all $\vec{v} \in V$.

(c) If $\vec{v}_1 \sim_W \vec{v}_2$ and $\vec{v}_2 \sim_W \vec{v}_3$ then $\vec{v}_1 \sim_W \vec{v}_3$, for all $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$.

Remark: We call such a relation an *equivalence relation*[12]. We can then define the *quotient set* V/W. For $\vec{v} \in V$, the *conjugacy class* of $\vec{v}$ is the set[13] $[v] := \{\vec{u} \in V | \vec{u} \sim_W \vec{v}\}$. The *set* V/W is the set of $\sim_W$ conjugacy classes in V.

(d) Draw some conjugacy classes of the relation $\sim_W$ on $\mathbb{R}^2$ for $W = Span(\begin{pmatrix} 1 \\ 2 \end{pmatrix}) \subset \mathbb{R}^2$.

We equip V/W with the structure of a vector space as follows:

- $[\vec{v}] + [\vec{u}] = [\vec{v} + \vec{u}]$.
- $a[\vec{v}] = [a\vec{v}]$.

(Harder) Show that

(e) The addition and scalar multiplication operations above are well defined[14].

(f) That these operations make V/W into a vector space.

(12) (Harder) A Better approach to row space: If we have a linear function $f : V \to W$ between two finite dimensional vector spaces V and W, then we can describe the column space $CS(f) = Range(f) \subset W$, and the Null space $N(f) = f^{-1}(\vec{0})$ intrinsically. That is to say they do not depend on a choice of basis.

Defining the row space is already difficult. A basis $\mathcal{B} := \{\vec{v}_1, ..., \vec{v}_n\}$ of V gives us a linear map[15] $\Phi_{\mathcal{B}} : \mathbb{R}^n \to V$, $\Phi_{\mathcal{B}}((a_1, ..., a_n)^T) = a_1 v_1 + \vec{...} + a_n \vec{v}_n$. We can write a matrix A_f representing f with respect to the basis $\{\vec{v}_1, ..., \vec{v}_n\}$, and a basis $\{\vec{w}_1, ..., \vec{w}_n\}$ of W. The row space of this matrix $RS(A_f) \subset \mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$. Hence $\Phi_{\mathcal{B}}(RS(A_f))$ is a vector subspace of V.

(a) Show that $\Phi_{\mathcal{B}}(RS(A_f)) \subset V$ depends on the choice of bases[16] – that is to say it is not an intrinsic property of f (Note that *both* the map $\Phi_{\mathcal{B}}$, and the matrix A_f change as the basis $\mathcal{B}$ is changed). Ie. Give an example of a linear transformation f and two choices of bases for V and W such that for the matrices $A_{f,1}$, $A_{f,2}$ representing f in these bases, and the maps $\Phi_1, \Phi_2 : \mathbb{R}^n \to V$ corresponding to these bases, we have

$$\Phi_1(RS(A_{f,1})) \neq \Phi_2(RS(A_{f,2}))$$

as subspaces of V.

(b) Which changes of basis of V, W preserve $\Phi_1(RS(A_{f,1}))$?

The moral of the story is that instead of using the $RS(A_f)$, we should use $V/N(f)$, ($N(f) = Ker(f)$ is the Kernel of f, or the null space of f) which we call the *coimage* of f, and does not depend on a choice of basis.

(c) Show that[17] $V/N(f) \oplus N(f) \cong V$ (ie. that there is an isomorphism $V \to V/N(f) \oplus N(f)$).

[11] This is relation in the sense of "binary relation," which you can look up on e.g. wikipedia. The idea is that this is completely analogous to how we define (e.g.) "$\leq$"; we write $a \leq b$ if and only if $b - a$ is a non-negative real number. Similarly here we are defining a *relation between vectors* "$\sim_W$", and we write $\vec{v}_1 \sim_W \vec{v}_2$ if and only if $\vec{v_1} - \vec{v_2} \in W$.

[12] You may wish to look up equivalence relation and quotient set on wikipedia or otherwise.

[13] *Warning:* The notation of square brackets is now overloaded as we are using it for both matrices and for conjugacy classes.

[14] That is to say the definition makes sense, and does define a way to add vectors, and scale by scalars. This is not obvious because suppose that $[v_1] = [v_2]$. Then $[\vec{v}_1] + [\vec{u}] = [\vec{v}_2] + [\vec{u}]$. Hence one requirement for this definition of addition to make sense it needs to be the case that $[\vec{v}_1 + \vec{u}] = [\vec{v}_2 + \vec{u}]$.

[15] In fact a linear isomorphism.

[16] Strictly speaking it only depends on the choice of basis of V

[17] See question 2(b) for the notation $\oplus$.

(d) Given a basis give an isomorphism $RS(A_f) \to V/N(f)$.

Remark: For the first of these you should notice that there is no natural choice of isomorphism. So the most "natural" relation in the absence of an inner product is that the sequence

$$1 \to N(f) \to V \to V/N(f) \to 1$$

has the property that the image of each arrow is the kernel of the next arrow (as we move from left to right).