MATH 312 SUMMER 2019: ASSIGNMENT 3

Due: Tuesday 30th July, beginning of class..

Up to 5 questions can be handed in on Thursday, and I will return them (by email) with comments by Saturday at the latest. This may be useful for Midterm 2 preparation.

Note: Answers must be fully explained (unless otherwise specified), using full sentences for full credit. You are encouraged to work on these questions with your classmates, however please write down all people you discussed problems with on your assignment, and *write up all solutions on your own.* Asking these questions on online discussion boards is not acceptable.

(1) (Question 7 from Assignment 2) Consider graph 1 from question 12 of Assignment 1. How many paths of length 100 are there between any two vertices?

(2) The Lucas numbers are a minor modification of the Fibonacci numbers. The Lucas numbers $\{L_n\}$ (for n a non-negative integer) are defined by the recurrence:

$$L_n = \begin{cases} 2, & \text{if } n = 0. \\ 1, & \text{if } n = 1. \\ L_{n-1} + L_{n-2}, & \text{otherwise.} \end{cases}$$

Find an equation for the k^{th} Lucas number, that does not involve any of the other Lucas numbers. *Hint:* Write (L_n, L_{n-1}) as a function of (L_{n-1}, L_{n-2}).

(3) Consider the Markov chain with transition matrix[1]

$$T = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Note: This can be seen as a small modification of the betting example from the lecture.

Denote the states corresponding to the vector $(1, 0, 0, 0)^T$ as state 1, the state corresponding to $(0, 1, 0, 0)^T$ as state 2, the state corresponding to $(0, 0, 1, 0)^T$ as state 3, and the remaining state as state 4. If we start in state 2, what is the probability the system will end up in state 1 and what is the probability we will end up in state 4?

(4) Suppose we have a (real) matrix A such that $\vec{x}^T A\vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$. Show that all eigenvalues of A are non-negative real numbers.

(5) Find the SVD decomposition of

$$A = \begin{pmatrix} \frac{3}{\sqrt{2}} & -1 & 1 \\ \frac{3}{\sqrt{2}} & 1 & -1 \end{pmatrix}$$

(6) Do a principal component analysis of the distribution of 3 variables, where our observed data is (Todo: This needs a computer)

- $(3, 1, 3)$,
- $(3, 3, 1)$,
- $(2, 2, 2)$,
- $(2, 2, 3)$,
- $(1, 1, 1)$.

(7) We want to fit a linear equation $w = ax + by + cz$ to the following observed data points (x, y, z, w): (Todo: this needs a computer)

[1] Some sources would use T^T rather than T and would act on probability vectors on the right.

- $(-3, -6, -9, -28)$
- $(9, -6, -3, 17)$
- $(5, -10, -1, -52)$
- $(1, -2, -11, 23)$.

Find the least squares plane of best fit to these data points. That is find a, b, c such that $\sum_{i=1}^{4}(w_i - ax_i - by_i - cz_i)^2$ is minimized.

(8) Let A be an $n \times m$ matrix. Let A^+ be the Moore–Penrose Pseudoinverse.
   (a) Suppose that $Ker(A) = \{\vec{0}\}$ (recall that the kernel of A, $(Ker(A))$, is another name for the null space of A). Show that $A^+A = Id_m$ (the $m \times m$ identity matrix).
   (b) Suppose that $Im(A) = \mathbb{R}^n$ (recall that $Im(A)$ denotes the image of A, which we also call the column space of A). Show that $AA^+ = Id_n$.
   (c) Describe the kernel of A^+, image of A^+, row space of A^+, and the null space of $(A^+)^T$ in terms of the kernels, images, null spaces and row spaces of A and A^T.

(9) We define a *Normed Vector space* to be a vector space V over the real or complex numbers together with a map (which we call the norm) $\|-\| : V \to \mathbb{R}$ with the following properties (We write $\|-\|(\vec{x})$ as $\|\vec{x}\|$):
- $\|\vec{x}\| \geq 0$ for all $\vec{x} \in V$, with $\|\vec{x}\| = 0$ if and only if $\vec{x} = 0$.
- $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$ for any vector $\vec{x} \in V$, and any scalar α. Note that for α complex $|\alpha|$ is the positive square root of $\alpha\overline{\alpha}$.
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ for all $\vec{x}, \vec{y} \in V$.

Show that:
   (a) For an inner product space $(V, \langle -, - \rangle)$, we get a normed vector space $(V, \|-\|)$, where $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$.
   (b) (Harder) If V and W are (finite dimensional) inner product spaces then $Hom(V, W)$ (See assignment 2) is made a normed vector space by the norm

$$\|T\| := \max_{\vec{v} \in V, \vec{v} \neq 0} \frac{\|Tv\|}{\|v\|}$$

where the norms on the right hand side of the above equation come from the inner products on V and W.

(10) We denote by $\kappa(T)$ the condition number[2] of T, for an invertible linear function T between inner product spaces.
   (a) Show that $\kappa(ST) \leq \kappa(S)\kappa(T)$ ($S : V \to W$, $T : W \to U$ invertible linear transformations between inner product spaces V, W, and U).
   (b) Write down a matrix with a condition number of 2.
   (c) Write down a matrix with a condition number of 1.
   (d) (Harder) Describe all 2×2 matrices with a condition number[3] of 1.

(11) Consider graph 1 from question 12 of assignment 2. Consider the Markov process where the states correspond to nodes of this graph, and at each time step we either
- We stay in the same state with probability 0.85.
- If we are at a node with i outgoing edges, we move along one of these edges with probability $\frac{0.15}{i}$.

   (a) What is the transition matrix T for this Markov process?
   (b) Using a computer or otherwise calculate T^{256}.

[2]Recall that $\kappa(T) = \|T\|\|T^{-1}\|$, where we use the norm of part (b) of the previous question of this assignment.

[3]By the condition number of a matrix we mean the condition number of its associated linear transformation.