CHAPTER TWO

Number Contemplation

2.1 Counting

2.2 Numerical Patterns in Nature

2.3 Prime Cuts of Numbers

2.4 Crazy Clocks and Checking Out Bars

2.5 Public Secret Codes and How to Become a Spy

2.6 The Irrational Side of Numbers

2.7 Get Real
Life is full of numbers. The moment we were born, our parents probably noted the time, our weight, our length, possibly our width, and most important, counted our toes. Numbers accompany us throughout life. We use numbers to measure our age, keep track of how much we owe on our charge cards, measure our wealth. (Notice how negative numbers may sneak in on that last one if we’re in debt.) In fact, does any aspect of daily life not involve counting or measurement? Numbers are a part of human life.

In this chapter we explore the notion of number. Just as numbers play a fundamental role in our daily lives, they also play a fundamental role within the realm of mathematics. We will come to see the richness of numbers and delve into their surprising traits. Some collections of numbers fit so well together that they actually lead to notions of aesthetics and beauty, whereas other numbers are so important that they may be viewed as basic building blocks. Relationships among numbers turn out to have powerful implications in our modern world, for example, within the context of secret codes and the Internet. Exploring the numbers we know leads us to discover whole new worlds of numbers beyond our everyday understanding. Within this expanded universe of number, many simple questions are still unanswered—mystery remains.

One of the main goals of this book is to illustrate methods of investigating the unknown, wherever in life it occurs. In this chapter we highlight some guiding principles and strategies of inquiry by using them to develop ideas about numbers.

Remember that an intellectual journey does not always begin with clear definitions and a list of facts. It often involves stumbling, experimenting, and searching for patterns. By investigating the world of numbers we encounter powerful modes of thought and analysis that can profoundly influence our daily lives.
We begin with the numbers we first learned as children: 1, 2, 3, 4, . . . (The “ . . . ” indicate that there are more, but we don’t have enough room to list them.) These numbers are so natural to us they are actually called natural numbers. These numbers are familiar, but often familiar ideas lead to surprising outcomes, as we will soon see.

The most basic use of numbers is counting, and we will begin by just counting approximately. That is, we’ll consider the power and the limitations of making rough estimates. In a way, this is the weakest possible use we can make of numbers, and yet we will still find some interesting outcomes. So let’s just have some fun with plain old counting.
Quantitative Estimation

One powerful technique for increasing our understanding of the world is to move from qualitative thinking to quantitative thinking whenever possible. Some people still count: “1, 2, 3, many.” Counting in that fashion is effective for a simple existence but does not cut it in a world of trillion-dollar debts and gigabytes of hard-drive storage. In our modern world there are practical differences between thousands, millions, billions, and trillions. Some collections are easy to count exactly because there are so few things in them: the schools in the Big Ten Conference, the collection of letters you’ve written home in the past month, and the clean underwear in your dorm room. Other collections are more difficult to count exactly—such as the grains of sand in the Sahara Desert, the stars in the sky, and the hairs on your roommate’s body. Let’s look more closely at this last example.

It would be difficult, awkward, and frankly just plain weird to count the number of hairs on your roommate’s body. Without undertaking that perverse task, we nevertheless pose the following.

Question Do there exist two nonbald people on the planet who have exactly the same number of hairs on their bodies?

It appears that we cannot answer this question since we don’t know (and don’t intend to find out) the body-hair counts for anyone. But can we estimate body-hair counts well enough to get some idea of what those numbers might be? In particular, can we at least figure out a number that we could state with confidence is larger than the number of hairs on the body of any person on Earth?

How Hairy Are We?

Let’s take the direct approach to this body-hair business. One of the authors counted the number of hairs on a 1/4-inch × 1/4-inch square area on his scalp and counted about 100 hairs—that’s roughly 1600 hairs per square inch. From this modest follicle count, we can confidently say that no person on Earth has as many as 16,000 hairs in any square inch anywhere on his or her body. The author is about 72 inches tall and 32 inches around. If the author were a perfect cylinder, he would have 72-inch × 32-inch or about 2300 square inches of skin on the sides and about another 200 square inches for the top of his head and soles of his feet, for a total of 2500 square inches of skin. Since the author is not actually a perfect cylinder (he has, for example,
a neck), 2500 square inches is an overestimate of his skin area. There are people who are taller and bigger than this author, but certainly there is no one on this planet who has 10 times as much skin as this author. Therefore, no body on Earth will have more than 25,000 square inches of skin. We already agreed that each square inch can have no more than 16,000 hairs on it. Thus we deduce that no person on this planet can have more than 400 million hairs on his or her body.

How Many Are We?
An almanac or a Web site would tell us that there are more than 6.8 billion people on this planet. Given this information, can we answer our question: Do there exist two nonbald people on the planet who have exactly the same number of hairs on their bodies? We urge you to think about this question and try to answer it before reading on.

Why Many People Are Equally Hairy
There are more than 6.8 billion people on Earth, but each person has many fewer than 400 million hairs on his or her body. Could it be that no two people have the same number of body hairs? What would that mean? It would mean that each of the 6.8 billion people would have a different number of body hairs. But we know that the number of body hairs on each person is less than 400 million. So, there are less than 400 million different possible body-hair numbers. Therefore, not all 6.8 billion people can have different body-hair counts.

Suppose we have 400 million rooms—each numbered in order. Suppose each person did know his or her body-hair count, and we asked each person in the world to go into the room whose number is equal to his or her body-hair number. Could everyone go into a different room? Of course not! We have more than 6 billion people and only 400 million room choices—some room or rooms must have more than one person. In other words, there definitely exist two people, in fact many people, who have the same number of body hairs.

By using some simple estimates, we have been able to answer a question that first appeared unanswerable. The surprising twist is that in this case a rough estimate led to a conclusion about an exact equality. However, there are limitations to our analysis. For example, we are unable to name two specific people who have the same body-hair counts even though we know they are out there.

The Power of Reasoning
In spite of the silliness of our hair-raising question we see the power of reasoned analysis. We were faced with a question that on first inspection appeared unanswerable, but through creative thought we were able to
crack it. When we are first faced with a new question or problem, the ultimate path of logical reasoning is often hidden from sight. When we try, think, fail, think some more, and try some more, we finally discover a path.

We solved the hairy-body question, but that question in itself is not of great value. However, once we have succeeded in resolving an issue, it is worthwhile to isolate the approach we used, because the method of thought may turn out to be far more important than the problem it solved. In this case, the key to answering our question was the realization that there are more people on the planet than there are body hairs on any individual’s body. This type of reasoning is known as the Pigeonhole principle.

If we have an antique desk with slots for envelopes (known as pigeonholes) like the one shown, and we have more envelopes than slots, then certainly some slot must contain at least two envelopes. This Pigeonhole principle is a simple idea, but it is a useful tool for drawing conclusions when the size of a collection exceeds the number of possible variations of some distinguishing trait.

Once we understand the Pigeonhole principle, we become conscious of something that has always been around us—we see it everywhere. For example, in a large swim meet, some pairs of swimmers will get exactly the same times to the tenths of a second. Some days more than 100 people will die in car wrecks. With each breath, we breathe an atom that Einstein breathed before us. Each person will arrive at work during the exact same minute many times during his or her life. Many trees have the same number of leaves. Many people get the same SAT score.

**Number Personalities**

The natural numbers 1, 2, 3, . . . , besides being useful in counting, have captured the imagination of people around the world from different cultures and different eras. The study of natural numbers began several thousand years ago and continues to this day. Mathematicians who are intrigued by numbers come to know them individually. In the eye of the mathematician, individual numbers have their own personalities—unique characteristics and distinctions from other numbers. In subsequent sections of this chapter, we will discover some intriguing properties of numbers and uncover their nuances. For now, however, we wish to share a story that captures the human side of mathematicians. Of course, mathematicians, like people in other professions, display a large range of personalities, but this true story of Ramanujan and Hardy depicts almost a caricature of the “pure” mathematician. It illustrates part of the mythology of mathematics and provides insight into the personality of an extraordinary mathematician.
Ramanujan and Hardy

One of the most romantic tales in the history of the human exploration of numbers involves the life and work of the Indian mathematician Srinivasa Ramanujan. Practically isolated from the world of academics, libraries, and mathematicians, Ramanujan made amazing discoveries about natural numbers.

In 1913, Ramanujan wrote to the great English mathematician G. H. Hardy at Cambridge University, describing his work. Hardy immediately recognized that Ramanujan was a unique jewel in the world of mathematics, because Ramanujan had not been taught the standard ways to think about numbers and thus was not biased by the rigid structure of a traditional education; yet he was clearly a mathematical genius. Since the pure nature of mathematics transcends languages, customs, and even formal training, Ramanujan’s imaginative explorations have since given mathematicians everywhere an exciting and truly unique perspective on numbers.

Ramanujan loved numbers as his friends, and found each to be a distinct wonder. A famous illustration of Ramanujan’s deep connection with numbers is the story of Hardy’s visit to Ramanujan in a hospital. Hardy later recounted the incident: “I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one and that I hoped it was not an unfavorable omen. ‘No,’ he replied, ‘it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.’” Notice that, indeed, \(1729 = 12^3 + 1^3\), and also \(1729 = 10^3 + 9^3\).

This interaction of two mathematicians on such an abstract plane even during serious illness is poignant. They clearly thought each number was worthy of special consideration. To affirm their special regard for each number, we now demonstrate conclusively that every natural number is interesting.

The Intrigue of Numbers.

Every natural number is interesting.

Proof that Natural Numbers Are Interesting

Let’s first consider the number 1. Certainly 1 is interesting, because it is the first natural number and it is the only number with this property: If we pick any number and then multiply it by 1, the answer is the original number we picked. So, we agree that the first natural number is interesting.
Let us now consider the number 2. Well, 2 is the first even number, and that is certainly interesting—and, if that weren’t enough, remember that 2 is the smallest number of people required to make a baby. Thus, we know that 2 is genuinely interesting.

We now consider the number 3. Is 3 interesting? Well, there are only two possibilities: Either 3 is interesting, or 3 is not interesting. Let us suppose that 3 is not interesting. Then notice that 3 has a spectacular property: It is the smallest natural number that is not interesting—which is certainly an interesting property! Thus we see that 3 is, after all, quite interesting.

Knowing now that 1, 2, and 3 are all interesting, we can make an analogous argument for 4 or any other number. In fact, suppose now that \( k \) is a certain natural number with the property that the first \( k \) natural numbers are all interesting. That is, 1, 2, 3, \ldots, \( k \) are all interesting. We know this fact is true if \( k \) is 1, and, in fact, it is true for larger values of \( k \) as well (2, 3, and 4, for example).

We now consider the very next natural number: \( k + 1 \). Is \( k + 1 \) interesting? Suppose it were not interesting. Then it would be the smallest natural number that is not interesting (all the smaller natural numbers would be known to be interesting). Well, that's certainly interesting! Thus, \( k + 1 \) must be interesting, too. Since we have shown that there can be no smallest uninteresting number, we must conclude that every natural number is interesting.

Our proof employs a logical “domino effect” to establish the validity of the theorem. This proof technique is known as mathematical induction and is used to prove many important mathematical results (most of which are not nearly as silly as the one we just established here).

A Look Back

Natural numbers are the natural place to begin our journey. This deceptively simple collection of numbers plays a significant role in our lives. We can understand our world more deeply by moving from qualitative to quantitative understanding. Counting and estimating, together with the Pigeonhole principle, lead to surprising insights about everyday events. Natural numbers help us understand our world, but they also constitute a world of their own. The whimsical assertion that every natural number is interesting foreshadows our quest to discover the variety and individual essence of these numbers.

Our strategy for understanding the richness of numbers was to start with the most basic and familiar use for numbers—counting. Looking carefully at the simple and the familiar is a powerful technique for creating and discovering new ideas.


**Mindscapes Invitations to Further Thought**

*In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (ExH) have expanded hints at the back of the book. Mindscapes marked (S) have solutions.*

I. Developing Ideas

1. **Muchos mangos.** You inherit a large crate of mangos. The top layer has 18 mangos. Peering through the cracks in the side of the crate, you estimate there are five layers of mangos inside. About how many mangos did you inherit?

2. **Packing balls.** Your best friend is about to turn 21 and you want to send him a box full of Ping-Pong balls. You have a square box measuring 12 inches on each side and you wonder how many Ping-Pong balls would fit inside. Suppose you have just enough balls to cover the bottom of the box in a single layer. How could you estimate the number that would fill the box?

3. **Alternative rock.** You have an empty CD rack consisting of five shelves and you just bought five totally kickin’ CDs. Can each CD go on a different shelf? What if you had six new CDs?

4. **The Byrds.** You have 16 new CDs to put on your empty five-shelf CD rack. Can you place the CDs so that each shelf contains three or fewer CDs? Can you arrange them so that each shelf contains exactly three?

5. **Fort hebi rds.** Explain the Pigeonhole principle.

II. Solidifying Ideas

6. **Treasure chest (ExH).** Someone offers to give you a million dollars ($1,000,000) in one-dollar ($1) bills. To receive the money, you must lie down; the million one-dollar bills will be placed on your stomach.
If you keep them on your stomach for 10 minutes, the money is yours! Do you accept the offer? Carefully explain your answer using quantitative reasoning.

7. **Order please.** Order the following numbers from smallest to largest:
- number of telephones on the planet
- number of honest members of Congress
- number of people
- number of grains of sand
- number of states in the United States
- number of cars

8. **Penny for your thoughts (H).** Two thousand years ago, a noble Arabian king wished to reward his minister of science. Although the modest minister resisted any reward from the king, the king finally forced him to state a desired reward. Impishly the minister said that he would be content with the following token: “Let us take a checkerboard. On the first square I would be most grateful if you would place one piece of gold. Then on the next square twice as much as before, thus placing two pieces, and on each subsequent square, placing twice as many pieces of gold as in the previous square. I would be most content with all the gold that is on the board once your majesty has finished.” This sounded extremely reasonable, and the king agreed. Given that there are 64 squares on a checkerboard, roughly how many pieces of gold did the king have to give to our “modest” minister of science? Why did the king have him executed?

9. **Twenty-nine is fine.** Find the most interesting property you can, unrelated to size, that the number 29 has and that the number 27 does not have.

10. **Perfect numbers.** The only natural numbers that divide evenly into 6, other than 6 itself, are 1, 2, and 3. Notice that the sum of all those numbers equals the original number 6 (1 + 2 + 3 = 6). What is the next number that has the property of equaling the sum of all the natural numbers other than itself that divide evenly into it? Such numbers are called perfect numbers. No one knows whether or not there are infinitely many perfect numbers. In fact, no one knows whether there are any odd perfect numbers. These two unsolved mysteries are examples of long-standing open questions in the theory of numbers.

11. **Many fold (S).** Suppose you were able to take a large piece of paper of ordinary thickness and fold it in half 50 times. What would the height of the folded paper be? Would it be less than a foot? About one yard? As long as a street block? As tall as the Empire State Building? Taller than Mount Everest?

12. **Only one cake.** Suppose we had a room filled with 370 people. Will there be at least two people who have the same birthday?

13. **For the birds.** Years ago, before overnight delivery services and e-mail, people would send messages by carrier pigeon and would keep an ample supply of pigeons in pigeonholes on their rooftops.
Suppose you have a certain number of pigeons, let’s say $P$ of them, but you have only $P - 1$ pigeonholes. If every pigeon must be kept in a hole, what can you conclude? How does the principle we discussed in this section relate to this question?

14. **Sock hop (ExH).** You have 10 pairs of socks, five black and five blue, but they are not paired up. Instead, they are all mixed up in a drawer. It’s early in the morning, and you don’t want to turn on the lights in your dark room. How many socks must you pull out to guarantee that you have a pair of one color? How many must you pull out to have two good pairs (each pair is the same color)? How many must you pull out to be certain you have a pair of black socks?

15. **The last one.** Here is a game to be played with natural numbers. You start with any number. If the number is even, you divide it by 2. If the number is odd, you triple it (multiply it by 3), and then add 1. Now you repeat the process with this new number. Keep going. You win (and stop) if you get to 1. For example, if we start with 17, we would have:

17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 → we see a 1, so we win!

Play four rounds of this game starting with the numbers 19, 11, 22, and 30. Do you think you will always win no matter what number you start with? No one knows the answer!

### III. Cr eating Ne w I deas

16. **See the three.** What proportion of the first 1000 natural numbers have a 3 somewhere in them? For example, 135, 403, and 339 all contain a 3, whereas 402, 677, and 8 do not.

17. **See the three II (H).** What proportion of the first 10,000 natural numbers contain a 3?

18. **See the three III.** Explain why almost all million-digit numbers contain a 3.

19. **Commuting.** One hundred people in your neighborhood always drive to work between 7:30 and 8:00 a.m. and arrive 30 minutes later. Why must two people always arrive at work at the same time, within a minute?

20. **RIP (S).** The Earth has more than 6.8 billion people and almost no one lives 100 years. Suppose this longevity fact remains true. How do you know that some year soon, more than 50 million people will die?
IV. Further Challenges

21. Say the sequence. The following are the first few terms in a sequence. Can you figure out the next few terms and describe how to find all the terms in the sequence?

1
11
21
1211
111221
312211

...

22. Lemonade. You want to buy a new car, and you know the model you want. The model has three options, each one of which you can either take or not take, and you have a choice of four colors. So far 100,000 cars of this model have been sold. What is the largest number of cars that you can guarantee to have the same color and the same options as each other?

V. In Your Own Words

23. With a group of folks. In a small group, discuss and work through the reasoning for why there are two people on Earth having the same number of hairs on their bodies. After your discussion, write a brief narrative describing your analysis and conclusion in your own words.
Discovering the Beauty of the Fibonacci Numbers

There is no inquiry which is not finally reducible to a question of Numbers; for there is none which may not be conceived of as consisting in the determination of quantities by each other, according to certain relations.

AUGUSTE COMTE

Often when we see beauty in nature, we are subconsciously sensing hidden order—order that itself has an independent richness. Thus we stop and smell the roses—or, more accurately, count the daisies. In the previous section, we contented ourselves with estimation, whereas here we move to exact counting. The example of counting daisies is an illustration of discovering numerical patterns in nature through direct observation. The pattern we find in the daisy appears elsewhere in nature and also gives rise to issues of aesthetics that touch such diverse fields as architecture and painting. We begin our investigation, however, firmly rooted in nature.

Have you ever examined a daisy? Sure, you’ve picked off the white petals one at a time while thinking: “Loves me . . . loves me not,” but have you ever taken a good hard look at what’s left once you’ve finished plucking? A close inspection of the yellow in the middle of the daisy reveals unexpected structure and intrigue. Specifically, the yellow area contains clusters of spirals coiling out from the center. If we examine the flower closely, we see that there are, in fact, two sets of spirals—a clockwise
set and a counterclockwise set. These two sets of spirals interlock to produce a hypnotic interplay of helical form.

Interlocking spirals abound in nature. The cone flower and the sunflower both display nature’s signature of dual, locking spirals. Flowers are not the only place in nature where spirals occur. A pinecone’s exterior is composed of two sets of interlocking spirals. The rough and prickly facade of a pineapple also contains two collections of spirals.

**Be Specific: Count**

In our observations we should not be content with general impressions. Instead, we move toward the specific. In this case we ponder the quantitative quandary: How many spirals are there? An approximate count is: lots. Is the number of clockwise spirals the same as the number of counterclockwise spirals? You can physically verify that the pinecone has 5 spirals in one direction and 8 in the other. The pineapple has 8 and 13. The daisy and cone flower both have 21 and 34. The sunflower has a staggering 55 and 89. In each case, we observe that the number of spirals in one direction is nearly twice as great as the number of spirals in the opposite direction. Listing all those numbers in order we see

5, 8, 13, 21, 34, 55, 89.

Is there any pattern or structure to these numbers?

Suppose we were given just the first two numbers, 5 and 8, on that list of spiral counts. How could we use these two numbers to build the next number? How can we always generate the next number on our list?
We note that 13 is simply 5 plus 8, whereas 21, in turn, is 8 plus 13. Notice that this pattern continues. What number would come after 89? Given this pattern, what number should come before 5? How about before that? How about before that? And before that?

**Leonardo’s Legacy: the Fibonacci Sequence**

The rule for generating successive numbers in the sequence is to add up the previous two terms. So the next number on the list would be \(55 + 89 = 144\).

Through spiral counts, nature appears to be generating a sequence of numbers with a definite pattern that begins

\[
1 \; 1 \; 2 \; 3 \; 5 \; 8 \; 13 \; 21 \; 34 \; 55 \; 89 \; 144 \ldots
\]

This sequence is called the Fibonacci sequence, named after the mathematician Leonardo of Pisa (better known as Fibonacci—a shortened form of Filius Bonacci, *son of Bonacci*), who studied it in the 13th century. After seeing this surprising pattern, we hope you feel compelled to count for yourself the spirals in the previous pictures of flowers. In fact, you may now be compelled to count the spirals on a pineapple every time you go to the grocery store.

Why do the numbers of spirals always seem to be consecutive terms in this list of numbers? The answer involves issues of growth and packing. The yellow florets in the daisy begin as small buds in the center of the plant. As the plant grows, the young buds move away from the center toward a location where they have the most room to grow—that is, in the direction that is least populated by older buds. If one simulates this tendency of the buds to find the largest open area as a model of growth on a computer, then the spiral counts in the geometrical pattern so constructed will appear in our list of numbers. The Fibonacci numbers are an illustration of surprising and beautiful patterns in nature. The fact that nature and number patterns reflect each other is indeed a fascinating concept.

A powerful method for finding new patterns is to take the abstract patterns that we directly observe and look at them by themselves. In this case, let’s move beyond the vegetable origins of the Fibonacci numbers and just think about the Fibonacci sequence as an interesting entity in its own right. We conduct this investigation with the expectation that interesting relationships that we find among Fibonacci numbers may also be represented in our lives.

**Fibonacci Neighbors**

We observed that flowers, pinecones, and pineapples all display consecutive pairs of Fibonacci numbers. These observations point to some natural bond between adjacent Fibonacci numbers. In each case, the number of spirals in one direction was not quite twice as great as the number of spirals in the other direction. Perhaps we can find richer structure and develop a deeper understanding of the Fibonacci numbers by moving from an estimate (“not quite twice”) to a precise value. So, let’s measure the relative
size of each Fibonacci number in comparison to the next one. We measure the relative size of one number in comparison to another by considering their ratio—that is, by dividing one of the numbers into the other. Here we list the quotients of adjacent Fibonacci numbers. Use a calculator to compute the last three terms in the chart to the right.

What do we notice about these answers? In the display to the right, notice that the pairs of Fibonacci numbers are getting larger and larger in size. But what about their relative sizes?

**Converging Quotients**

The relative sizes—that is, the quotients of consecutive Fibonacci numbers in the right column—seem to oscillate. They get bigger, then smaller, then bigger, then smaller, but they are apparently becoming increasingly close to one another and are converging toward some intermediate value. What is the exact value for the target number toward which these ratios are heading?

To find it, let’s look again at those quotients of Fibonacci numbers, but this time let’s write those fractions in a different way. Looking at the same information from a different vantage point often leads to insight. If we’re careful with the arithmetic and remember the rule for building the Fibonacci numbers, we will uncover an unusual pattern of 1’s. Notice how we use the pattern of 1’s from one quotient to produce a pattern of 1’s for the next quotient. Each step below involves the facts that \( \frac{a}{b} = \frac{1}{(b/a)} \) and that each Fibonacci number can be written as the sum of the previous two.

\[
\begin{align*}
1 \quad & = \quad 1 \\
\frac{2}{1} \quad & = \quad \frac{1+1}{1} = 1 + \frac{1}{1} \\
\frac{3}{2} \quad & = \quad \frac{2+1}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{1 + \frac{1}{1}} \\
\frac{5}{3} \quad & = \quad \frac{3+2}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{1}{3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} \\
\frac{8}{5} \quad & = \quad \frac{5+3}{5} = \frac{5}{5} + \frac{3}{5} = 1 + \frac{1}{5} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Fraction of Adjacent Fibonacci Numbers</th>
<th>Decimal Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{1} )</td>
<td>1.0</td>
</tr>
<tr>
<td>( \frac{2}{1} )</td>
<td>2.0</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>1.5</td>
</tr>
<tr>
<td>( \frac{5}{3} )</td>
<td>1.666...</td>
</tr>
<tr>
<td>( \frac{8}{5} )</td>
<td>1.6</td>
</tr>
<tr>
<td>( \frac{13}{8} )</td>
<td>1.625</td>
</tr>
<tr>
<td>( \frac{21}{13} )</td>
<td>1.6153...</td>
</tr>
<tr>
<td>( \frac{34}{21} )</td>
<td>1.6190...</td>
</tr>
<tr>
<td>( \frac{55}{34} )</td>
<td>1.6176...</td>
</tr>
<tr>
<td>( \frac{89}{55} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{144}{89} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{233}{144} )</td>
<td></td>
</tr>
</tbody>
</table>
Let’s look at what we’re doing. Replacing the top Fibonacci number in the numerator of our fraction by the sum of the previous two Fibonacci numbers allows us to see a pattern. For example,

\[
\frac{233}{144} = \frac{144 + 89}{144} = 1 + \frac{89}{144} = 1 + \frac{1}{\frac{144}{89}}.
\]

Now notice that 144/89 would be the previous fraction on our list. So 144/89 would have already been written as a long fraction of 1’s.

If we continue this process we see that the ratio of any two adjacent Fibonacci numbers is a number that looks like this:

\[
1 + \frac{1}{1 + \frac{1}{1 + \cdots}} + 1.
\]

**Unending 1’s**

As we compute the quotient of ever larger Fibonacci numbers, the ratios head toward the strange expression: \(1 + 1/(1 + 1/(1 + \ldots))\) in which we mean by “\(\ldots\)” that this fraction never ends, the quotient goes on forever. Only in mathematics can we create something that is truly unending. Let’s give this unending number a name. We call this number \(\varphi\) (\(\varphi\) is the lowercase Greek letter phi, and in our journey through geometry a few chapters from now, we’ll find out why it is called \(\varphi\)—stay tuned). So we see that

\[
\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}}.
\]

where the dots mean this process goes on forever. Remember, our goal is to figure out what number the quotients of consecutive Fibonacci numbers approach, and we now see that they approach \(\varphi\). But what exactly does \(\varphi\) equal? Since it’s described in such an interesting form—as an infinitely long fraction containing only 1’s—it seems impossible to know the precise value of \(\varphi\), or is it possible? Right now, the answer is not clear. So let’s look for some pattern within that exotic expression for \(\varphi\).

Before attempting to answer the preceding question, we first ask a warm-up question: We are going to write \(\varphi\) out again; however, this time, notice that we have placed a frame around part of \(\varphi\). Here is our question: What does the number inside the frame equal?
\[ \phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}} \]

The answer is: The number in the frame is \( \phi \) again. Why? Well, suppose we were just shown the number inside the frame without any of that other stuff around it. We’d look at that new number and realize that the 1’s go on forever in the same pattern as before, and thus that number is just \( \phi \). Stay with this picture until you see the idea behind it. Therefore, we just discovered that

\[ \phi = 1 + \frac{1}{\phi}. \]

**Solving for \( \phi \)**

Now we have an equation involving just \( \phi \), and this will allow us to solve for the exact value of \( \phi \). First, we can subtract 1 from both sides to get

\[ \phi - 1 = \frac{1}{\phi}. \]

Multiplying through by \( \phi \) we get

\[ \phi^2 - \phi = 1 \]

or just

\[ \phi^2 - \phi - 1 = 0. \]

This “quadratic equation” can be solved using the quadratic formula, which implies that

\[ \phi = \frac{1 \pm \sqrt{5}}{2}. \]

But since \( \phi \) is bigger than 1, we must have

\[ \phi = \frac{1 + \sqrt{5}}{2}. \]

Using a calculator, express \( (1 + \sqrt{5}) / 2 \) as a decimal and compare it with the data from our previous calculator experimentation on the quotients of consecutive Fibonacci numbers. Well, there we have it—our goal was to find the exact value of \( \phi \), and through a process of observation and thought we succeeded.

**The Golden Ratio**

At the moment, the only interesting features of \( \phi \) we have seen are that it is the fixed value the quotients of consecutive Fibonacci numbers approach and that it can be expressed in a remarkable way as an endless
“fraction within a fraction within a fraction…,” just using 1’s. We started with simple observations of flowers and pinecones. We saw a numerical pattern among our observations. The pattern led us to the number \( \frac{1 + \sqrt{5}}{2} \).

The number \( \varphi = \frac{1 + \sqrt{5}}{2} \) is called the Golden Ratio and, besides its connection with nature’s spirals, it captures the proportions of some especially pleasing shapes in art, architecture, and geometry. Just to foreshadow what is to come when we revisit the Golden Ratio in the geometry chapter, here is a question: What are the proportions of the most attractive rectangle? In other words, when someone says “rectangle” to you, and you think of a shape, what is it? Light some scented candles, put on a Yanni CD, close your eyes, and dream about the most attractive and pleasing rectangle you can imagine. Once that image is etched in your mind, open your eyes, put out the candles, and pick from the four choices below the rectangle that you think is most representative of that magical rectangle dancing in your mind.

Many people feel that the second rectangle from the left is the most aesthetically pleasing—the one that captures the notion of “rectangle-ness.” That rectangle is called the Golden Rectangle, and we will examine it in detail in Chapter 4. The ratio of the dimensions of the sides of the Golden Rectangle is a number that we will see is rich with intrigue. If we divide the length of the longer side by the length of the shorter side, we again come upon \( \varphi \): the Golden Ratio. A 3-inch \( \times \) 5-inch index card is close to being a Golden Rectangle. Notice that its dimensions, 3 and 5, are consecutive Fibonacci numbers. In the geometry chapter we will consider the (at times controversial) aesthetic issues involving \( \varphi \) and make some interesting connections between the Fibonacci numbers and the Golden Rectangle in art.

**To Be or Not to Be Fibonacci**

After finding Fibonacci numbers hidden in the spirals of nature, it saddens us to realize that not all numbers are Fibonacci. However, we are delighted to announce that in fact every natural number is a neat sum of Fibonacci numbers. In particular, every natural number is either a Fibonacci number or it is expressible uniquely as a sum of nonconsecutive Fibonacci numbers. Here is one way to find the sum:

1. Write down a natural number.
2. Find the largest Fibonacci number that does not exceed your number. That Fibonacci number is the first term in your sum.
3. Subtract that Fibonacci number from your number and look at this new number.

4. Find the largest Fibonacci number that does not exceed this new number. That Fibonacci number is the second number in your sum.

5. Continue this process.

For example, consider the number 38. The largest Fibonacci number not exceeding 38 is 34. So consider $38 - 34 = 4$. The largest Fibonacci number not exceeding 4 is 3, and $4 - 3 = 1$, which is a Fibonacci number. Therefore, $38 = 34 + 3 + 1$. Similarly, we can build any natural number just by adding Fibonacci numbers in this manner. In one sense, Fibonacci numbers are building blocks for the natural numbers through addition.

**Fun and Games with Fibonacci**

Fibonacci numbers not only appear in nature; they can also be used to accumulate wealth. (Moral: Math pays.) We can see this moral for ourselves in a game called *Fibonacci nim*, which is played with two people. All we need is a pile of sticks (toothpicks, straws, or even pennies will do). Person One moves first by taking any number of sticks (at least one but not all) away from the pile. After Person One moves, it is Person Two’s move, and the moves continue to alternate between them. Each person (after the first move) may take away as many sticks as he or she wishes; the only restriction is that he or she must take at least one stick but no more than two times the number of sticks the previous person took. The player who takes the last stick wins the game.

Suppose we start with ten sticks, and Person One removes three sticks, leaving seven. Now Person Two may take any number of the remaining sticks from one to six (six is two times the number Person One took). Suppose Person Two removes five, leaving two in the pile. Now Person One is permitted to take any number of sticks from one to ten ($10 = 2 \times 5$), but because there are only two sticks left, Person One takes the two sticks and wins. Play Fibonacci nim with various friends and with different numbers of starting sticks. Get a feel for the game and its rules—but don’t wager quite yet.

If we are careful and use the Fibonacci numbers, we can always win. Here is how. First, we make sure that the initial number of sticks we start with is not a Fibonacci number. Now we must be Person One, and we find some poor soul to be Person Two. If we play it just right, we will always win. The secret is to write the number of sticks in the pile as a sum of nonconsecutive Fibonacci numbers. Figure out the smallest Fibonacci number occurring in the sum, and remove that many sticks from the pile on the first move. Now it is your luckless opponent’s turn. No matter what he or she does, we will repeat the preceding procedure. That is, once he or she is done, we count the number of sticks in the pile, express the number as a sum of nonconsecutive Fibonacci numbers, and then
remove the number of sticks that equals the smallest Fibonacci number in the sum. It is a fact that, no matter what our poor opponent does, we will always be able to remove that number of sticks without breaking the rules. Experiment with this game and try it. Wager at will—or not.

**A Look Back**

We define the Fibonacci numbers successively by starting with 1, 1, and then adding the previous two terms to get the next term. These numbers are rich with structure and appear in nature. The numbers of clockwise and counterclockwise spirals in flowers and other plants are consecutive Fibonacci numbers. The ratio of consecutive Fibonacci numbers approaches the Golden Ratio, a number with especially pleasing proportions. While not all numbers are Fibonacci, every natural number can be expressed as the sum of distinct, nonconsecutive Fibonacci numbers.

The story of Fibonacci numbers is a story of pattern. As we look at the world, we can often see order, structure, and pattern. The order we see provides a mental concept that we can then explore on its own. As we discover relationships in the pattern, we frequently find that those same relationships refer back to the world in some intriguing way.

---

Understand simple things deeply.

---

**Mindscapes Invitations to Further Thought**

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (ExH) have expanded hints at the back of the book. Mindscapes marked (S) have solutions.

1. **De veloping Id eas**

1. **Fifteen Fibonacci.** List the first 15 Fibonacci numbers.

2. **Born φ.** What is the precise number that the symbol φ represents? What sequence of numbers approaches φ?

3. **Tons o f ones.** Verify that $1 + \frac{1}{1 + \frac{1}{1}}$ equals 3/2.
4. **Twos and threes.** Simplify the quantities $2 + \frac{2}{2}$ and $3 + \frac{3}{3}$.

5. **The family of $\varphi$.** Solve the following equations for $x$:
   
   $x = 2 + \frac{1}{x}, x = 3 + \frac{1}{x}$.

### II. Solifying Ideas

6. **Baby bunnies.** This question gave the Fibonacci sequence its name. It was posed and answered by Leonardo of Pisa, better known as Fibonacci.

   Suppose we have a pair of baby rabbits: one male and one female. Let us assume that rabbits cannot reproduce until they are one month old and that they have a one-month gestation period. Once they start reproducing, they produce a pair of bunnies each month (one of each sex). Assuming that no pair ever dies, how many pairs of rabbits will exist in a particular month?

   During the first month, the bunnies grow into rabbits. After two months, they are the proud parents of a pair of bunnies. There will now be two pairs of rabbits: the original, mature pair and a new pair of bunnies. The next month, the original pair produces another pair of bunnies, but the new pair of bunnies is unable to reproduce until the following month. Thus we have:

<table>
<thead>
<tr>
<th>Time in Months</th>
<th>Start</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Pairs</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   Continue to fill in this chart and search for a pattern. Here is a suggestion: Draw a family tree to keep track of the offspring.
We’ll use the symbol $F_1$ to stand for the first Fibonacci number, $F_2$ for the second Fibonacci number, $F_3$ for the third Fibonacci number, and so forth. So $F_1 = 1$, and $F_2 = 1$, and, therefore, $F_3 = F_2 + F_1 = 2$, and $F_4 = F_3 + F_2 = 3$, and so on. In other words, we write $F_n$ for the $n$th Fibonacci number where $n$ represents any natural number; for example, we denote the 10th Fibonacci number as $F_{10}$ and hence we have $F_{10} = 55$. So, the rule for generating the next Fibonacci number by adding up the previous two can now be stated, in general, symbolically as:

$$F_n = F_{n-1} + F_{n-2}.$$

7. **Discovering Fibonacci relationships (S).** By experimenting with numerous examples in search of a pattern, determine a simple formula for $(F_{n+1})^2 + (F_n)^2$; that is, a formula for the sum of the squares of two consecutive Fibonacci numbers. (See Mindscape II.6 for a description of the notation $F_n$.)

8. **Discovering more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a simple formula for $(F_{n+1})^2 - (F_{n-1})^2$; that is, a formula for the difference of the squares of two Fibonacci numbers. (See Mindscape II.6 for a description of the notation $F_n$.)

9. **Late bloomers (ExH).** Suppose we start with one pair of baby rabbits, and again they create a new pair every month, but this time let’s suppose that it takes two months before a pair of bunnies is mature enough to reproduce. Make a table for the first 10 months, indicating how many pairs there would be at the end of each month. Do you see a pattern? Describe a general formula for generating the sequence of rabbit-pair counts.

10. **A new start.** Suppose we build a sequence of numbers using the method of adding the previous two numbers to build the next one. This time, however, suppose our first two numbers are 2 and 1. Generate the first 15 terms. This sequence is called the *Lucas sequence* and is written as $L_1, L_2, L_3, \ldots$. Compute the quotients of consecutive terms of the Lucas sequence as we did with the Fibonacci numbers. What number do these quotients approach? What role do the initial values play in determining what number the quotients approach? Try two other first terms and generate a sequence. What do the quotients approach?

11. **Discovering Lucas relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for $L_{n-1} + L_{n+1}$; that is, a formula for the sum of a Lucas number and the Lucas number that comes after the next one. (*Hint: The answer will not be a Lucas number. See Mindscape II.10.*)
12. **Still more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for \( F_{n-1} + F_{n+1} \); that is, a formula for the sum of a Fibonacci number and the Fibonacci number that comes after the next one. (See Mindscape II.6 for a description of the notation \( F_n \).) *(Hint: The answer will not be a Fibonacci number. Try Mindscape II.10 first.)*

13. **Even more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for \( F_{n+2} - F_{n-2} \); that is, a formula for the difference between a Fibonacci number and the fourth Fibonacci number before it. (See Mindscape II.6 for a description of the notation \( F_n \).) *(Hint: The answer will not be a Fibonacci number. Try Mindscape II.10 first.)*

14. **Discovering Fibonacci and Lucas relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for \( F_n + L_n \); that is, a formula for the sum of a Fibonacci number and the corresponding Lucas number. (See Mindscape II.6 for a description of the notation \( F_n \).)

15. **The enlarging area paradox (S).** The square below has sides equal to 8 (a Fibonacci number) and thus has area equal to \( 8 \times 8 = 64 \). The square can be cut up into four pieces whereby the short sides have lengths 3 and 5, as illustrated. (You will find these pieces in the kit that accompanied your book.) Now use those pieces to construct a rectangle having base 13 and height 5. The area of this rectangle is \( 13 \times 5 = 65 \). So by moving around the four pieces, we increased the area by 1 unit! Using the puzzle pieces from your kit, do the experiment and then explain this impossible feat. Show this puzzle to your friends and record their reactions.

16. **Sum of Fibonacci (H).** Express each of the following natural numbers as a sum of distinct, nonconsecutive Fibonacci numbers: 52, 143, 13, 88.

17. **Some more sums.** Express each of the following natural numbers as a sum of distinct, nonconsecutive Fibonacci numbers: 43, 90, 2000, 609.

18. **Fibonacci nim: The first move.** Suppose you are about to begin a game of Fibonacci nim. You start with 50 sticks. What is your first move?
19. **Fibonacci nim: The first move II.** Suppose you are about to begin a game of Fibonacci nim. You start with 100 sticks. What is your first move?

20. **Fibonacci nim: The first move III.** Suppose you are about to begin a game of Fibonacci nim. You start with 500 sticks. What is your first move?

21. **Fibonacci nim: The next move.** Suppose you are playing a round of Fibonacci nim with a friend. The game begins with 15 sticks. You start by removing two sticks; your friend then takes four. How many sticks should you take next to win?

22. **Fibonacci nim: The next move II.** Suppose you are playing a round of Fibonacci nim with a friend. The game begins with 50 sticks. You start by removing three sticks; your friend then takes five; you then take eight; your friend then takes ten. How many sticks should you take next to win?

23. **Fibonacci nim: The next move III.** Suppose you are playing a round of Fibonacci nim with a friend. The game begins with 90 sticks. You start by removing one stick; your friend then takes two; you take three; your friend takes six; you take two; your friend takes one; you take two; your friend takes four; you take one, and then your friend takes two. How many sticks should you take next to win?

24. **Beat your friend.** Play Fibonacci nim with a friend: Explain the rules, but do not reveal the secret to winning. Use 20 sticks to start. Play carefully, and beat your friend. Play again with another (non-Fibonacci) number of sticks to start. Record the number of sticks removed at each stage of each game. Finally, reveal the secret strategy and record your friend’s reaction.

25. **Beat another friend.** Play Fibonacci nim with another friend: Explain the rules, but do not reveal the secret to winning. Use 30 sticks to start. Play carefully and beat your friend. Play again with another (non-Fibonacci) number of sticks to start. Record the number of sticks removed at each stage of each game. Finally, reveal the secret strategy and record your friend’s reaction.

**III. Creating New Ideas**

26. **Discovering still more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for \( F_{n+1} \times F_{n-1} - (F_n)^2 \); that is, a formula for the product of a Fibonacci number and the Fibonacci number that comes after the next one, minus the square of the Fibonacci number in between them. (*Hint:* The answer will be different depending on whether \( n \) is even or odd. Consider examples of different cases separately.)

27. **Finding factors (S).** By experimenting with numerous examples, find a way to factor \( F_{2n} \) into the product of two natural numbers that
are from famous sequences. That is, consider every other Fibonacci number starting with the second 1 in the sequence, and factor each in an interesting way. Discover a pattern. (Hint: Mindscape II.10 may be relevant.)

28. The rabbits rest. Suppose we have a pair of baby rabbits—one male and one female. As before, the rabbits cannot reproduce until they are one month old. Once they start reproducing, they produce a pair of bunnies (one of each sex) each month. Now, however, let us assume that each pair dies after three months, immediately after giving birth. Create a chart showing how many pairs we have after each month from the start through month nine.

29. Digging up Fibonacci roots. Using the square root key on a calculator, evaluate each number in the top row and record the answer in the bottom row.

<table>
<thead>
<tr>
<th>Number</th>
<th>( \sqrt{\frac{F_1}{F_2}} )</th>
<th>( \sqrt{\frac{F_2}{F_3}} )</th>
<th>( \sqrt{\frac{F_3}{F_4}} )</th>
<th>( \sqrt{\frac{F_5}{F_6}} )</th>
<th>( \sqrt{\frac{F_7}{F_8}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computed Value</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Looking at the chart, make a guess as to what special number \( \sqrt{\frac{F_{n+2}}{F_n}} \) approaches as \( n \) gets larger and larger.

30. Tribonacci. Let’s start with the numbers 0, 0, 1, and generate future numbers in our sequence by adding up the previous three numbers. Write out the first 15 terms in this sequence, starting with the first 1. Use a calculator to evaluate the value of the quotients of consecutive terms (dividing the smaller term into the larger one). Do the quotients seem to be approaching a fixed number?

31. Fibonacci follies. Suppose you are playing a round of Fibonacci nim with a friend. You start with 15 sticks. You start by removing two sticks; your friend then takes one; you take two; your friend takes one. What should your next move be? Can you make it without breaking the rules of the game? Did you make a mistake at some point? If so, where?

32. Fibonacci follies II. Suppose you are playing a round of Fibonacci nim with a friend. You start with 35 sticks. You start by removing one stick; your friend then takes two; you take three; your friend takes six; you take three; your friend takes two. What should your next move be? Can you make it without breaking the rules of the game? Did you make a mistake at some point? If so, where?

33. Fibonacci follies III. Suppose you are playing a round of Fibonacci nim with a friend. You start with 21 sticks. You start by removing one stick; your friend then takes two. What should your next move be? Can you make it without breaking the rules of the game? What went wrong?
34. **A big fib (ExH).** Suppose we have a natural number that is not a Fibonacci number—let’s call it \(N\). Suppose that \(F\) is the largest Fibonacci number that does not exceed \(N\). Show that the number \(N - F\) must be smaller than the Fibonacci number that comes right before \(F\).

35. **Decomposing naturals (H).** Use the result of Mindscape III.34, together with the notion of systematically reducing a problem to a smaller problem, to show that every natural number can be expressed as a sum of distinct, nonconsecutive Fibonacci numbers.

**IV. Further Challenges**

36. **How big is it?** Is it possible for a Fibonacci number greater than 2 to be exactly twice as big as the Fibonacci number immediately preceding it? Explain why or why not. What would your answer be if we removed the phrase “greater than 2”?

37. **Too small.** Suppose we have a natural number that is not a Fibonacci number—let’s call it \(N\). Let’s write \(F\) for the largest Fibonacci number that does not exceed \(N\). Show that it is impossible to have a sum of two distinct Fibonacci numbers each less than \(F\) add up to \(N\).

38. **Beyond Fibonacci.** Suppose we create a new sequence of natural numbers starting with 0 and 1. Only this time, instead of adding the two previous terms to get the next one, let’s generate the next term by adding 2 times the previous term to the term before it. In other words: \(G_{n+1} = 2G_n + G_{n-1}\). Such a sequence is called a *generalized Fibonacci sequence*. Write out the first 15 terms in this generalized Fibonacci sequence. Adapt the methods that were used in this section to figure out that the quotient of consecutive Fibonacci numbers approaches \((1 + \sqrt{5}) / 2\) to discover the exact number that \(G_{n+1}/G_n\) approaches as \(n\) gets large.

39. **Generalized sums.** Let \(G_n\) be the generalized Fibonacci sequence defined in Mindscape IV.38. Can every natural number be expressed as the sum of distinct, nonconsecutive generalized Fibonacci numbers? Show why, or give several counterexamples. What if you were allowed to use consecutive generalized Fibonacci numbers? Do you think you could do it then? Illustrate your hunch with four or five specific examples.

40. **It’s hip to be square (H).** Adapt the methods of this section to prove that the numbers \(\sqrt{F_{n+2}/F_n}\) approach \(\varphi\) as \(n\) gets larger and larger. (Here, \(F_n\) stands for the usual \(n\)th Fibonacci number. See Mindscape I I.6.)
V. In Your Own Words

41. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

42. **With a group of folks.** In a small group, discuss and work through the reasoning for how the quotients of consecutive Fibonacci numbers approach the Golden Ratio. After your discussion, write a brief narrative describing the rationale in your own words.

43. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.
2.3 Prime Cuts of Numbers

How the Prime Numbers Are the Building Blocks of All Natural Numbers

... number is merely the product of our mind.

KARL FRIEDRICH GAUSS

The natural numbers, 1, 2, 3, . . . , help us describe and understand our world. They in turn form their own invisible world filled with abstract relationships, some of which can be revealed through simple addition and multiplication. These basic operations lead to subtle insights about our familiar numbers.

Our strategy for uncovering the structure of the natural numbers is to break down complex objects and ideas into their fundamental components. This simple yet powerful technique recurs frequently throughout this book, and throughout our lives. As we become accustomed to using this strategy, we will see that complicated situations are often best analyzed first by investigating the building blocks of an idea or an object and then by understanding how these building blocks combine to create a complex whole. The natural numbers are a good arena for observing this principle in action.

Charles Demuth, I Saw the Figure 5 in Gold, 1928
Building Blocks
What are the fundamental components of the natural numbers? How can we follow the suggestion of breaking down numbers into smaller components?

There are many ways to express large natural numbers in terms of smaller ones. For example, we might first think of addition: Every natural number can be constructed by just adding \(1 + 1 + 1 + 1 + \cdots + 1\) enough times. This method demonstrates perhaps the most fundamental feature of natural numbers: They are simply a sequence of counting numbers, each successive one bigger than its predecessor. However, this feature provides only a narrow way of distinguishing one natural number from another.

Divide and Conquer
How can one natural number be expressed as the product of smaller natural numbers? This innocent-sounding question leads to a vast field of interconnections among the natural numbers that mathematicians have been exploring for thousands of years. Our adventure begins by recalling the arithmetic from our youth and looking at it afresh.

One method of writing a natural number as a product of smaller ones is first to divide and then to see if there is a remainder. We were introduced to division a long time ago in third or fourth grade—we weren’t impressed. Somehow it paled in comparison to, say, recess. The basic reality of long division is that either it comes out even or there is a remainder. If the division comes out even, we then know that the smaller number divides evenly into the larger number and that our number can be factored. For example, 12 divided by 3 is 4, so \(3 \times 4 = 12\) and 3 and 4 are factors of 12. More generally, suppose that \(n\) and \(m\) are any two natural numbers. We say that \(n\) divides evenly into \(m\) if there is a natural number \(q\) such that

\[nq = m.\]

The integers \(n\) and \(q\) are called factors of \(m\).

If the division does not come out even, the remainder is less than the number we tried to divide. For example, 16 divided by 5 is 3 with a remainder of 1.

This whole collection of elementary school flashbacks can be summarized in a statement that sounds far more impressive than “long division,” namely, the Division Algorithm.
The Division Algorithm.

Suppose \( n \) and \( m \) are natural numbers. Then there exist unique numbers \( q \) (for quotient) and \( r \) (for remainder), that are either natural numbers or zero, such that

\[
m = nq + r \quad \text{and} \quad 0 \leq r \leq n - 1 \quad (r \text{ is greater than or equal to } 0 \text{ but less than or equal to } n - 1).
\]

Prime Time

Factoring a big number into smaller ones gives us some insights into the larger number. This method of breaking up natural numbers into their basic components leads us to the important notion of prime numbers.

There certainly are natural numbers that cannot be factored as the product of two smaller natural numbers. For example, 7 cannot be factored into two smaller natural numbers, nor can these: 2, 3, 5, 11, 13, 17, which, including 7, are the first seven such numbers (let’s ignore the number 1). These unfactorable numbers are called prime numbers. A prime number is a natural number greater than 1 that cannot be expressed as the product of two smaller natural numbers.

The prime numbers form the multiplicative building blocks for all natural numbers greater than 1. That is, every natural number greater than 1 is either prime or it can be expressed as a product of prime numbers.

The Prime Factorization of Natural Numbers.

Every natural number greater than 1 is either a prime number or it can be expressed as a product of prime numbers.

Let’s first look at a specific example and then see why the Prime Factorization of Natural Numbers is true for all natural numbers greater than 1.

Is 1386 prime? No, 1386 can be factored as

\[
1386 = 2 \times 693.
\]

Is 693 prime? No. It can be factored as

\[
693 = 3 \times 231.
\]

The number 3 is prime, but
231 = 3 \times 77.

So far we have

1386 = 2 \times 3 \times 3 \times 77.

Is 77 prime? No,

77 = 7 \times 11

but 7 and 11 are both primes. Thus we have

1386 = 2 \times 3 \times 3 \times 7 \times 11.

So, we have factored 1386 into a product of primes. This simple “divide and conquer” technique works for any number!

**Proof of the Prime Factorization of Natural Numbers**

Let \( n \) be any natural number greater than 1 that is not a prime. Then there must be a factorization of \( n \) into two smaller natural numbers both greater than 1 (why?), say \( n = a \times b \). We now look at \( a \) and \( b \). If both are primes, we end our proof, since \( n \) is equal to a product of two primes. If either \( a \) or \( b \) is not prime, then it can be factored into two smaller natural numbers, each greater than 1. If we continue in this manner with each of the factors, since the factors are getting smaller, we will find in a finite number of steps a factorization of \( n \) where all the factors are prime numbers. So, \( n \) is a product of primes, and our proof is complete.

**Really Big Numbers: We Can, but We Can’t**

Even though we know that every natural number can theoretically be factored, as a practical matter, factoring really large numbers is currently impossible. For example, suppose someone asks us to factor this long number:

2203019871351092560192573410934710923871024835618972658273
6458764058761984569183246501923847019237418726501826354109
8327401923847019234601982561092384760198237401923867450872
3640982367401982360192386092364019283470129834701293487102
93861023984610398710239847109238710293871923871093.

We might have some difficulty. In fact, the fastest computers in the world working full time would take centuries to factor this number. Our inability to factor really big numbers is the key to devising public key codes for sending private information over the Web and using e-mail and ATMs. We consider this modern application of prime numbers in Section 2.5, “Public Secret Codes and How to Become a Spy.”
What’s the Largest Prime?

The Prime Factorization of Natural Numbers reveals that prime numbers are the building blocks for the natural numbers; that is, to build any natural number, all we need to do is multiply the appropriate prime numbers together. Of course, whenever we are presented with building blocks, whether they are Lincoln Logs, Legos, elementary particles, or primes, the question arises: How many blocks do we have? In this case, we ask how many prime numbers are there? Are there more than a million primes? Are there more than a billion primes? Are there infinitely many primes?

Since there are infinitely many natural numbers, it seems reasonable to think that we would need bigger and bigger primes in order to build up the bigger and bigger natural numbers. Therefore, we might conclude that there are infinitely many primes. This argument, however, isn’t valid. Do you see why? It’s not valid because we can build natural numbers as large as we wish just by multiplying a couple of small prime numbers. For example, using just 2 and 3, we are able to make huge numbers:

\[ 45,349,632 = 2^6 \cdot 3^{11}. \]

The point is that, since we are allowed to use tons of 2’s and 3’s in our product, we can construct numbers as large as we wish. Therefore, at this moment it seems plausible that there may be only finitely many prime numbers, even though every large (and small) natural number is a product of primes.

Infinitely Many Primes

Although the previous argument was invalid, it turns out that there are infinitely many prime numbers, and thus there is no largest prime number. Euclid discovered the following famous valid proof more than 2000 years ago. It is beautiful in that the idea is clever and uncomplicated. In fact, on first inspection we may think there is some sleight of hand going on, as if some fast-talking salesman in a polyester plaid sports jacket is trying to sell us a used car—before we know it, we’re the not-so-proud owner of a 1973 Gremlin. But this is not the case. Think along with us as we develop this argument. The ideas fit together beautifully, and, if you stay with it, the argument will suddenly “click.” Let’s now examine one of the great triumphs of human reasoning.

Euclid (Ghent, Joos (Justus) van, Euclid, Palazzo Ducale, Urbino, Italy Scala/Art Resource, NY)

Euclid (Ghent, Joos (Justus) van, Euclid, Palazzo Ducale, Urbino, Italy Scala/Art Resource, NY)
The Strategy Behind the Proof

The strategy for proving that there are infinitely many prime numbers is to show that, for each and every given natural number, we can always find a prime number that is larger than that natural number. Since we can consider larger and larger numbers as our natural number, this claim would imply that there are larger and larger prime numbers. Thus we would show that there must be infinitely many primes, because we could find primes as large as we want without bound.

Before moving forward with the general idea of the proof, let’s illustrate the key ingredient with a specific example. Suppose we wanted to find a prime number that is greater than 4. How could we proceed? Well, we could just say “5” and be done with it, but that is not in the spirit of what we are trying to do. Our goal is to discover a method that can be generalized and used to find a prime number that exceeds an arbitrary natural number, not just the pathetically small number 4. So we seek a systematic means of finding a prime that exceeds, in this case, 4. What should we do? Our challenge is to:

1. find a number bigger than 4 that is not evenly divisible by 2;
2. find a number bigger than 4 that is not evenly divisible by 3; and
3. find a number bigger than 4 that is not evenly divisible by 4.

Each of these tasks individually is easy. To satisfy (1), we just pick an odd number. To satisfy (2), we just pick a number that has a remainder when we divide by 3. To satisfy (3), we just pick a number that has a remainder when we divide by 4. We now build a number that meets all those conditions simultaneously. Let’s call this new number \( N \) for new. Here is an \( N \) that meets all three conditions simultaneously:

\[
N = (1 \times 2 \times 3 \times 4) + 1.
\]

So, \( N \) is really just the number 25, but, since we are trying to discover a general strategy, let’s not think of \( N \) as merely 25 but instead think of \( N \) as the more impressive \( (1 \times 2 \times 3 \times 4) + 1 \).

We notice that \( N \) is definitely larger than 4. By the Prime Factorization of Natural Numbers, we know that \( N \) is either a prime number or a product of prime numbers. In the first possibility, if \( N \) is a prime number, then we have just found a prime number that is larger than 4—which was our goal. We now must consider the only other possibility: that \( N \) is not prime. If \( N \) is not prime, then \( N \) is a product of prime numbers. Let’s call one of those prime factors of \( N \) “OUR-PRIME.” So OUR-PRIME is a prime that divides evenly into \( N \).

Now what can we say about OUR-PRIME? Is OUR-PRIME equal to 2? Well, if we divide 2 into \( N \) we see that, since

\[
N = 2 \times (1 \times 3 \times 4) + 1,
\]
we have a remainder of 1 when 2 is divided into \( N \). Therefore, 2 does not divide evenly into \( N \), and so \( \text{OUR-PRIME} \) is not 2. Is \( \text{OUR-PRIME} \) equal to 3? No, for the same reason:

\[
N = 3 \times (1 \times 2 \times 4) + 1,
\]

so we get a remainder of 1 when 3 is divided into \( N \); therefore, 3 does not divide evenly into \( N \) and hence is not a factor of \( N \). Likewise, we will get a remainder of 1 when 4 is divided into \( N \), and, therefore, 4 is not a factor of \( N \). So 2, 3, and 4 are not factors of \( N \). Hence we conclude that all the factors of \( N \) must be larger than 4 (there are no factors of \( N \) that are 4 or smaller). But that means that, since \( \text{OUR-PRIME} \) is a prime that is a factor of \( N \), \( \text{OUR-PRIME} \) must be a prime number greater than 4. Therefore, we have just found a prime number greater than 4. Mission accomplished!

Following this same strategy, show that there must be a prime greater than 5. Can you use the preceding method to show there must be a prime greater than 10,000,000? Try it now.

**Finally, the Proof**

Now let’s use the method we developed in the specific example to prove our theorem in general. Remember that we wish to demonstrate that, for any particular natural number, there is a prime number that exceeds that particular number. Let \( m \) represent an arbitrary natural number. Our goal now is to show that there is a prime number that exceeds \( m \). To accomplish this lofty quest, we will, just as before, construct a new number using all the numbers from 1 to \( m \). We’ll call this new number \( N \) (for new number) and define it to be 1 plus the product of all the natural numbers from 1 to \( m \)—in other words (or more accurately in other symbols):

\[
N = (1 \times 2 \times 3 \times 4 \times \cdots \times m) + 1.
\]

It is fairly easy to see that \( N \) is larger than \( m \). By the Prime Factorization of Natural Numbers we know that there are only two possibilities for \( N \): Either \( N \) is prime or \( N \) is a product of primes. If the first possibility is true, then we have found what we wanted since \( N \) is larger than \( m \). But if it is not true, then we must consider the more challenging possibility: that \( N \) is a product of prime numbers.

If \( N \) is a product of primes, then we can choose one of those prime factors and call it \( \text{BIG-PRIME} \). So, \( \text{BIG-PRIME} \) is a prime factor of \( N \). Let’s now try to pin down the value of \( \text{BIG-PRIME} \). We’ll start with the small.

Does \( \text{BIG-PRIME} \) equal 2? Well, if we divide 2 into \( N \) we see that, since

\[
N = 2 \times (1 \times 3 \times 4 \times \cdots \times m) + 1,
\]

we have a remainder of 1 when 2 is divided into \( N \). Therefore, 2 does not divide evenly into \( N \), and so \( \text{BIG-PRIME} \) cannot equal 2.
Does $BIG-PRIME$ equal 3? No, for the same reason:

$$N = 3 \times (1 \times 2 \times 4 \times \cdots \times m) + 1,$$

so we get a remainder of 1 when 3 is divided into $N$. In fact, what is the remainder when any number from 2 to $m$ is divided into $N$? The remainder is always 1 by the same reasoning that we used for 2 and 3.

Okay, so we see that none of the numbers from 2 through $m$ divides evenly into $N$. That fact means that none of the numbers from 2 through $m$ is a factor of $N$. Therefore, what can be said about the size of the factor $BIG-PRIME$? Answer: It must be BIG since we know that $N$ has no factors between 2 and $m$. Hence any factor of $N$ must be larger than $m$. Therefore, $BIG-PRIME$ is a prime number that is larger than $m$.

Well, we did it! We just showed that there is a prime that exceeds $m$. Since this procedure works for any value of $m$, this argument shows that there are arbitrarily large prime numbers. Therefore, we must have infinitely many prime numbers, and we have completed the proof.

**The Clever Part of the Proof**

In the proof, each step by itself isn’t too hard, but the entire argument, taken as a whole, is subtle. What is the most clever part of the proof? In other words, where is the most imagination required? Which step in the argument would have been hardest to think up on your own?

We believe that the most ingenious part of the proof is the idea of constructing the auxiliary number $N$ (one more than the product of all the numbers from 1 to $m$). Once we have the idea of considering that $N$, we can finish the proof. But it took creativity and contemplation to arrive at that choice of $N$. We might well say to ourselves, “Gee, I wouldn’t have thought of making up that $N$.” Generally, slick proofs such as this are arrived at only after many attempts and false starts—just as Euclid no doubt experienced before he thought of this proof. Very few people can understand arguments of this type on first inspection, but once we can hold the whole proof in our minds, we will regard it as straightforward and persuasive and appreciate its aesthetic beauty. Ingenuity is at the heart of creative mathematical reasoning, and therein lies the power of mathematical thought.

**Prime Demographics**

Now that we know for sure that there are infinitely many prime numbers, we wonder how the primes are distributed among the natural numbers. Is there some pattern to their distribution? There are infinitely many primes, but how rare are they among the numbers? What proportion of the natural numbers are prime numbers? Half? A third? To explore these questions, let’s start by looking at the natural numbers and the primes among them. Here are the first few with the primes printed in bold:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, ...
Out of the first 24 natural numbers, nine are primes. We see that \( \frac{9}{24} = 0.375 \) of the first 24 natural numbers are primes—that’s just a little over one-third. Extrapolating from this observation would we guess that just over one-third of all natural numbers are prime numbers? We could try an experiment; namely, we could continue to list the natural numbers and find the proportions of primes and see whether that proportion remains about one-third of the total number. If we do this experiment, we will learn an important lesson in life: Don’t be too hasty to generalize based on a small amount of evidence.

Before high-speed computers were available, calculating (or just estimating) the proportion of prime numbers in the natural numbers was a difficult task. In fact, years ago “computers” were people who did computations. Such people were amazingly accurate, but they required a great deal of time and dedication to accomplish what today’s electronic computers can do in seconds. An 18th-century Austrian arithmetician, J. P. Kulik, spent 20 years of his life creating, by hand, a table of the first 100 million primes. His table was never published, and sadly the volume containing the primes between 12,642,600 and 22,852,800 has disappeared.

Today, software computes the number of primes less than \( n \) for increasingly large values of \( n \), and the computer prints out the proportion: \( \frac{(\text{number of primes less than } n)}{n} \). Computers have no difficulty producing such a table for values of \( n \) up to the billions, trillions, and beyond. If we examine the results, we notice that the proportion of primes slowly goes downward. That is, the percentage of numbers less than a million that are prime is smaller than the percentage of numbers less than a thousand that are prime. The primes, in some sense, get sparser and sparser as the numbers get bigger and bigger.

**A Conjecture About Patterns**

In the early 1800s, Karl Friedrich Gauss (below left), one of the greatest mathematicians ever—known by many as the Prince of Mathematics—and A. M. Legendre (below right), another world-class mathematician, made an insightful observation about primes. They noticed that, even though primes do not appear to occur in any predictable pattern, the proportion of primes is related to the so-called natural logarithm—a function relating to exponents that we may or may not remember from our school daze.

Years ago, one needed to interpolate huge tables to find the logarithm. Today, we have scientific calculators that compute logarithms instantly and painlessly. Get out a scientific calculator and look for the LN key. Hit the
LN key and then type “3.” You should see 1.09861…. We encourage you to try some natural-logarithm experiments on your calculator. How does the size of the natural logarithm of a number compare with the size of the number itself?

Gauss and Legendre conjectured that the proportion of the number of primes among the first $n$ natural numbers is approximately $1/\ln(n)$. The following chart, constructed with the aid of a computer (over which Gauss and Legendre would drool), shows the number of primes up to $n$, the proportions of primes, and a comparison with $1/\ln(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Number of Primes up to $n$</th>
<th>Proportion of Primes up to $n$ (Number of Primes $\leq n$)</th>
<th>$1/\ln(n)$</th>
<th>Proportion $- 1/\ln(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>0.4</td>
<td>0.43429…</td>
<td>-0.03429…</td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td>0.25</td>
<td>0.21714…</td>
<td>0.03285…</td>
</tr>
<tr>
<td>1000</td>
<td>168</td>
<td>0.168</td>
<td>0.14476…</td>
<td>0.02323…</td>
</tr>
<tr>
<td>10,000</td>
<td>1229</td>
<td>0.1229</td>
<td>0.10857…</td>
<td>0.01432…</td>
</tr>
<tr>
<td>100,000</td>
<td>9592</td>
<td>0.09592</td>
<td>0.08685…</td>
<td>0.00906…</td>
</tr>
<tr>
<td>1,000,000</td>
<td>78,498</td>
<td>0.078498</td>
<td>0.07238…</td>
<td>0.00611…</td>
</tr>
<tr>
<td>10,000,000</td>
<td>664,579</td>
<td>0.0664579</td>
<td>0.06204…</td>
<td>0.00441…</td>
</tr>
<tr>
<td>100,000,000</td>
<td>5,761,455</td>
<td>0.05761455</td>
<td>0.05428…</td>
<td>0.00332…</td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>50,847,534</td>
<td>0.050847534</td>
<td>0.04825…</td>
<td>0.00259…</td>
</tr>
</tbody>
</table>

Notice how the last column seems to be getting closer and closer to zero—that is, the proportion of primes in the first $n$ natural numbers is approximately $1/\ln(n)$, and the fraction $(\text{number of primes less than } n)/n$ is becoming increasingly closer to $1/\ln(n)$ the bigger $n$ gets. Our observations show that

$$(\text{number of primes up to } n)/n \text{ is roughly } 1/\ln(n).$$

Multiplying both of these quantities by $n$, it appears that

$$(\text{number of primes up to } n) \text{ is roughly } n/\ln(n).$$

These observations culminate in what is called the **Prime Number Theorem**, which gives the approximate number of primes less than or equal to $n$ as $n$ gets really big.

**The Prime Number Theorem.**

*As $n$ gets larger and larger, the number of prime numbers less than or equal to $n$ approaches $n/\ln(n)$.***

**The Proof Had to Wait**

Having sufficient insight into the nature of numbers to make such a conjecture is a tremendous testament to the intuition of Gauss and Legendre. But, even though they conjectured that the Prime Number
Theorem was true, they were unable to prove it. This famous conjecture remained unproved for nearly a century until, in 1896, two mathematicians, Hadamard and de la Vallee Poussin, simultaneously but independently constructed proofs of the Prime Number Theorem. That is, working on their own, they each found, in the same year, a proof of this extremely important and difficult result. The proof of this theorem is extremely long and uses the machinery of many branches of mathematics (including, of all things, imaginary numbers). This mathematical episode illustrates a hopeful and powerful fact about human thought: The human mind is capable of solving extremely difficult problems and answering truly hard questions. This observation should give us all hope for finding solutions to some of the many problems we face in our world today.

Some problems are solved in relatively short time periods; others take longer. In some cases long-standing mysteries concerning numbers are resolved only after centuries of vain attempts. When such stubborn questions are finally answered, the solutions represent a triumph for humanity. In the mid-1990s, with the diligence and drive of several generations of mathematicians, a 350-year-old problem was finally solved. Such a solution is a celebration of the human spirit and justly made the front page of The New York Times and the Wall Street Journal. This is the story of Fermat’s Last Theorem.

The Mostly True Story of Fermat’s Last Theorem

It was a dark and stormy night. Rain pelted the wavy window pane. Wind seeped in through cracks in the window frame. A candle flickered uncertainly in the cold breeze. The year was 1637.

Pierre de Fermat sat in the great chair of his library poring over the leather-bound Latin translation of the ancient tome of Diophantus’s *Arithmetica*. He concentrated intensely on the ideas that Diophantus had written 2000 years before. During those two millennia whole civilizations had risen and fallen, but the study of numbers bridged Fermat and Diophantus, spanning centuries and even transcending death itself.

As the night wore on, inchoate ideas, at first jumbled and ill-defined, began to weave a pattern in Fermat’s mind. He gasped as insight and understanding began to take shape. He was seeing how the parts fit together, how the numbers played on one another. Finally, it was clear to him.

On that fateful night in 1637 he wrote in the margin of Diophantus’s *Arithmetica* the statement that would make him famous forever. He wrote in Latin:

*It is impossible to write a cube as a sum of two cubes, a fourth power as a sum of two fourth powers, and, in general, any power beyond the second as a sum of two similar powers. For this, I have discovered a truly wondrous proof, but the margin is too small to contain it.*
He saw that, for any two natural numbers, if we take the first number raised to the third power, then add it to the second number raised to the third power, that sum will never equal a natural number raised to the third power. For example, suppose we consider the natural numbers 2 and 14. We note that

\[2^3 = 2 \times 2 \times 2 = 8, \quad 14^3 = 14 \times 14 \times 14 = 2744,\]

so

\[2^3 + 14^3 = 2752,\]

and there is no natural number that, when raised to the third power, equals 2752 (14^3 = 2744 is too small, and 15^3 = 3375 is too large).

What Fermat saw was that for any natural number greater than or equal to 3, let’s call that number \(n\), it is impossible to find three natural numbers, let’s call them \(x\), \(y\), and \(z\), such that \(x\) raised to the \(n\)th power (\(n\) \(x\)’s multiplied together) added to \(y\) raised to the \(n\)th power equals \(z\) raised to the \(n\)th power. In other words, in the language of mathematical symbols, for any natural number \(n\) greater than or equal to 3, there do not exist any natural numbers \(x\), \(y\), and \(z\) that satisfy the equation

\[x^n + y^n = z^n.\]

Fermat claimed that he saw a “wondrous proof,” but did he? Did he discover a correct proof, or did he deceive himself? No one knows for certain, because Fermat never wrote his “proof” down. In fact, Fermat was notorious for making such statements with usually little or no justification or proof. By the 1800s, all of Fermat’s statements had been resolved, all but the preceding one—his “last” one. This unproved assertion became known as Fermat’s Last Theorem.

We will never know whether Fermat had discovered a correct proof of his Last Theorem, but we do know one thing. He did not discover the proof that Andrew Wiles (right) of Princeton University produced in 1994, 357 years after Fermat wrote his tantalizing marginal note. If Fermat had somehow conceived of Wiles’s deep and complicated proof, he would not have written, “The margin of this book is not large enough to contain it.” He would have written, “The proof would require a moving van to carry it.” Wiles’s proof drew on entirely new branches of mathematics and incorporated ideas undreamed of in the 17th century.
Some of the greatest minds in mathematics have worked on Fermat’s Last Theorem. The statement of Fermat’s Last Theorem does not strike us as intrinsically important or interesting—it just states that a certain type of equation never can be solved with natural numbers. What is interesting and important are all the deep mathematical ideas that arose during attempts to prove Fermat’s Last Theorem, many of which led to new branches of mathematics. Although Fermat’s Last Theorem has not yet been used for practical purposes, the new theories developed to attack it turned out to be valuable in many practical technological advances. From an aesthetic perspective, it is difficult to determine which questions are more important than others. When problems resist attempts by the best mathematical minds over many years, the problems gain prestige. Fermat’s Last Theorem resisted all attacks for 357 years, but it finally succumbed.

Andrew Wiles’s complete proof of Fermat’s Last Theorem is over 130 pages long, and it relies on many important and difficult theorems, including some new theorems from geometry (although it appears surprising that geometry should play a role in solving this problem involving natural numbers). When mathematicians expose connections between seemingly disparate areas of mathematics, they feel an electric excitement and pleasure. In mathematics, as in nature, elements fit together and interrelate. As we will begin to discover, deep and rich connections weave their way through the various mathematical topics, forming the very fabric of truth.

**The Vast Unknown**

Many people think mathematics is a static, ancient body of facts, formulas, and techniques. In reality, much of it is a wondrous mystery with many questions unanswered and more still yet to be asked. Many people think we will soon know all there is to know. But this impression is not the case, even though such thinking has persisted throughout history.

*Everything that can be invented has been invented.*

CHARLES H. DUELL, COMMISSIONER
U.S. PATENT OFFICE, 1899

*We don’t know a millionth of one percent about anything.*

THOMAS EDISON
Human thought is an ever-expanding universe—especially in mathematics. We know a small amount, and our knowledge allows us to glimpse a small part of what we do not know. Vastly larger is our ignorance of what we do not know. An important shift in perspective on mathematics and other areas of human knowledge occurs when we move from the sense that we know most of the answers to the more accurate and comforting realization that we will not run out of mysteries.

So, after celebrating, as we have, some of the great mathematical achievements that were solved only after many decades of human creativity and thought, we close this section by gazing forward to questions that remain unsolved—open questions. From among the thousands of questions on which mathematicians are currently working, here are two famous ones about prime numbers that were posed hundreds of years ago and are still unsolved. Fermat’s Last Theorem has been conquered; but somehow the mathematical force is not ready to let go its hold and lets et wof all.

**The Twin Prime Question.**

*Are there infinitely many pairs of prime numbers that differ from one another by two? (11 and 13, 29 and 31, and 41 and 43 are examples of some such pairs.)*

**The Goldbach Question.**

*Can every positive, even number greater than 2 be written as the sum of two primes? (Pick some even numbers at random, and see whether you can write them each as a sum of two primes.)*

Computer analysis allows us to investigate a tremendous number of cases, but the results of such analyses do not provide ironclad proof for all cases.

We have seen how to decompose natural numbers into their fundamental building blocks, and we have discovered further mysteries and structures in this realm. Can we use these antique, abstract results about numbers in our modern lives? The amazing and perhaps surprising answer is a resounding yes!
A Look Back

The prime numbers are the basic multiplicative building blocks for natural numbers, since every natural number greater than 1 can be factored into primes. We can prove that there are infinitely many primes by showing that we can always find a prime number larger than any specified number. The strategy is to take the product of all numbers up to the specified number and then add 1. This new large integer must have all its prime factors greater than the original specified number.

The study of primes goes back to ancient times. Some questions remained unanswered for a long time before being resolved. And others remain unanswered.

We discovered proofs of the Prime Factorization of Natural Numbers and the Infinitude of Primes by carefully exploring specific examples and searching for patterns. Considering specific examples while thinking about the general case guided us to new discoveries.

Understanding a specific case well is a major step toward discovering a general principle.

Mindscapes Invitations to Further Thought

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (ExH) have expanded hints at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. Primal instincts. List the first 15 prime numbers.

2. Fear factor. Express each of the following numbers as a product of primes: 6, 24, 27, 35, 120.

3. Odd couple. If $n$ is an odd number greater than or equal to 3, can $n + 1$ ever be prime? What if $n$ equals 1?

4. Tower of power. The first four powers of 3 are $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, and $3^4 = 81$. Find the first 10 powers of 2. Find the first five powers of 5.

5. Compose a list. Give an infinite list of natural numbers that are not prime.
2.3 / Prime Cuts of Numbers

II. Solidifying Ideas

6. A silly start. What is the smallest number that looks prime but really isn’t?

7. Waiting for a nonprime. What is the smallest natural number \( n \), greater than 1, for which \((1 \times 2 \times 3 \times \cdots \times n) + 1\) is not prime?


10. Prime power. Is it possible for an extremely large prime to be expressed as a large integer raised to a very large power? Explain.

11. Nonprimes (ExH). Are there infinitely many natural numbers that are not prime? If so, prove it.

12. Prime test. Suppose you are given a number \( n \) and are told that 1 and the number \( n \) divide into \( n \). Does that mean \( n \) is prime? Explain.


14. Goldbach. Express the first 15 even numbers greater than 2 as the sum of two prime numbers.

15. Odd Goldbach (H). Can every odd number greater than 3 be written as the sum of two prime numbers? If so, prove it; if not, find the smallest counterexample and show that the number given is definitely not the sum of two primes.

16. Still the 1 (S). Consider the following sequence of natural numbers: 1111, 11111, 111111, 1111111, 11111111, \ldots. Are all these numbers prime? If not, can you describe infinitely many of these numbers that are definitely not prime?

17. Zeros and ones. Consider the following sequence of natural numbers made up of 0’s and 1’s: 11, 101, 1001, 10001, 100001, 1000001, 10000001, \ldots. Are all these numbers prime? If not, find the first such number that is not prime and express it as a product of prime numbers.

18. Zeros, ones, and threes. Consider the following sequence of natural numbers made up of 0’s, 1’s, and 3’s: 13, 103, 1003, 10003, 100003, 1000003, 10000003, \ldots. Are all these numbers prime? If not, find the first such number that is not prime and express it as a product of prime numbers.

19. A rough count. Using results discussed in this section, estimate the number of prime numbers that are less than \(10^{10}\).

20. Generating primes (H). Consider the list of numbers: \( n^2 + n + 17 \), where \( n \) first equals 1, then 2, 3, 4, 5, 6, \ldots. What is the smallest value
of \( n \) for which \( n^2 + n + 17 \) is not a prime number? (*Bonus:* Try this for \( n^2 - n + 41 \). You’ll see an amazingly long string of primes!)

21. **Generating primes II.** Consider the list of numbers: \( 2^n - 1 \), where \( n \) first equals 2, then 3, 4, 5, 6, \ldots . What is the smallest value of \( n \) for which \( 2^n - 1 \) is not a prime number?

22. **Floating in factors.** What is the smallest natural number that has three distinct prime factors in its factorization?

23. **Lucky 13 factor.** Suppose a certain number when divided by 13 yields a remainder of 7. What is the smallest number we would have to subtract from our original number to have a number with a factor of 13?

24. **Remainder reminder (S).** Suppose a certain number when divided by 13 yields a remainder of 7. If we add 22 to our original number, what is the remainder when this new number is divided by 13?

25. **Remainder roundup.** Suppose a certain number when divided by 91 yields a remainder of 52. If we add 103 to our original number, what is the remainder when this new number is divided by 7?

### III. Creating New Ideas

26. **Related remainders (H).** Suppose we have two numbers that both have the same remainder when divided by 57. If we subtract the two numbers, are there any numbers that we know will definitely divide evenly into this difference? What is the largest number that we are certain will divide into the difference? Use this observation to state a general principle about two numbers that have the same remainder when divided by another number.

27. **Prime differences.** Write out the first 15 primes all on one line. On the next line, underneath each pair, write the difference between the larger number and the smaller number in the pair. Under this line, below each pair of the previous line, write the difference between the smaller number and the larger number. Continue in this manner. Your “triangular” table should begin with:

\[
\begin{array}{cccccc}
2 & 3 & 5 & 7 & 11 & 13 & 17 & \ldots \\
1 & 2 & 2 & 4 & \ldots \\
1 & 0 & 2 & \ldots \\
& & & & & & & \text{and so on} \ldots
\end{array}
\]

Once your chart is made, imagine that all the primes were listed on that first line. What would you guess is the pattern for the sequence of numbers appearing in the first entry of each line? The actual answer is not known. It remains an open question! What do you think?
28. **Minus two.** Suppose we take a prime number greater than 3 and then subtract 2. Will this new number always be a prime? Explain. Are there infinitely many primes for which the answer to the question is yes? How does this last question relate to a famous open question?

29. **Prime neighbors.** Does there exist a number $n$ such that both $n$ and $n + 1$ are prime numbers? If so, find such an $n$; if not, show why not.

30. **Perfect squares.** A perfect square is a number that can be written as a natural number squared. The first few perfect squares are 1, 4, 9, 16, 25, 36. How many perfect squares are less than or equal to 36? How many are less than or equal to 144? In general, how many perfect squares are less than or equal to $n^2$? Using all these answers, estimate the number of perfect squares less than or equal to $N$. (*Hint:* Your estimate may involve square roots and should be the exact answer whenever $N$ is itself a perfect square.)

31. **Perfect squares versus primes.** Using a calculator or a computer, fill in the last two columns of the following chart.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Number of Primes up to $n$</th>
<th>Approximate Number of Perfect Squares up to $n$</th>
<th>Number of Primes/Number of Perfect Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>168</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10,000</td>
<td>1229</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100,000</td>
<td>9592</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000,000</td>
<td>78,498</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10,000,000</td>
<td>664,579</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100,000,000</td>
<td>5,761,455</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000,000,000</td>
<td>50,847,534</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Given the information found in your chart, what do you conclude about the proportion of prime numbers to perfect squares? Are prime numbers more or less common than perfect squares? Using the Prime Number Theorem, estimate the quotient of the number of primes up to $n$ divided by the number of perfect squares up to $n$. Use a computer or a graphing calculator to graph your answer. Does the graph confirm your original conjecture?

32. **Prime pairs.** Suppose that $p$ is a prime number greater than or equal to 3. Show that $p + 1$ cannot be a prime number.

33. **Remainder addition.** Let $A$ and $B$ be two natural numbers. Suppose that, when $A$ is divided by $n$, the remainder is $a$, and, when $B$ is divided by $n$, the remainder is $b$. How does the remainder when $A + B$ is divided by $n$ compare with the remainder when $a + b$ is divided by $n$? Try some specific examples first. Can you prove your answer?

34. **Remainder multiplication.** Let $A$ and $B$ be two natural numbers. Suppose that the remainder when $A$ is divided by $n$ is $a$, and the
remainder when $B$ is divided by $n$ is $b$. How does the remainder when $A \times B$ is divided by $n$ compare with the remainder when $a \times b$ is divided by $n$? Try some specific examples first. Can you prove your answer?

35. **A prime-free gap (S).** Find a run of six consecutive natural numbers, none of which is a prime number. (*Hint:* Prove that you can start with $[1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7] + 2$.)

**IV. Further Challenges**

36. **Prime-free gaps.** Using Mindscape III.35, show that, for a given number, there exists a run of that many consecutive natural numbers, none of which is a prime number.

37. **Three primes (ExH).** Prove that it is impossible to have three consecutive integers, all of which are prime.

38. **Prime plus three.** Prove that if you take any prime number greater than 11 and add 3 to it the sum is not prime.

39. **A small factor.** Prove that if a number greater than 1 is not a prime number, then it must have a prime factor less than or equal to its square root. (*Hint:* Suppose that all the factors are greater than its square root, and show that the product of any two of them must be larger than the original number.) This Mindscape shows that, to determine if a number is a prime, we have only to check divisibility of the primes up to the square root of the number.

40. **Prime products (H).** Suppose we make a number by taking a product of prime numbers and then adding the number 1—for example, $(2 \times 5 \times 17) + 1$. Compute the remainder when any of the primes used is divided into the number. Show that none of the primes used can divide evenly into the number. What can you conclude about the primes that divide evenly into the number? Can you use this line of reasoning to give another proof that there are infinitely many prime numbers?

**V. In Your Own Words**

41. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.
42. **With a group of folks.** In a small group, discuss and work through the proof that there are infinitely many primes. After your discussion, write a brief narrative describing the proof in your own words.

43. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.