Lecture 11 - Cartan Subalgebras

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1 Maximal Toral Subalgebras

A *toral subalgebra* of a Lie algebra $\mathfrak{g}$ is any subalgebra consisting entirely of abstractly semisimple elements.

Lemma 1.1 *If $\mathfrak{t} \subset \mathfrak{g}$ is any toral subalgebra, then $\mathfrak{t}$ is abelian.*

Pf. Assume $x \in \mathfrak{t}$ has $(ad\, x)|_{\mathfrak{t}} \neq 0$. Because x is semisimple, it is ad-simisimple, so there is some $y \in \mathfrak{t}$ with $(ad\, x)(y) = ay$. Note that

$$0 \;=\; (ad\, y)(ad\, x)(y) \;=\; -(ad\, y)^2 x. \tag{1}$$

However, y is also semisimple, so we can write

$$x \;=\; x_1 \,+\, \ldots \,+\, x_n \tag{2}$$

where $(ad\, y)(x_i) = \lambda_i x_i$. Therefore

$$0 \;=\; (ad\, y)^2 x \;=\; \sum_{i=1}^{n} \lambda_i^2 x_i \tag{3}$$

so that each $\lambda_i = 0$. However this contradicts

$$0 \;\neq\; ay \;=\; (ad\, x)y \;=\; -(ad\, y)x \;=\; \sum_{i=1}^{n} \lambda_i x_i. \tag{4}$$

□

If $\mathfrak{g}$ is a semisimple Lie algebra, then any maximal toral subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* or CSA for short. Caution: in the non-semisimple case, this is not the proper use of the term CSA.

Since $\mathfrak{h}$ is a commuting subalgebra, the action of $ad_{\mathfrak{g}}\mathfrak{h}$ is simultaneously diagonalizable. This means that $\mathfrak{g}$ has a complete root-space decomposition: there are finitely many linear functionals $\alpha \in \mathfrak{h}^* \triangleq \Lambda$, $\alpha : \mathfrak{h} \to \mathbb{F}$ so that

$$\mathfrak{g} = \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha \tag{5}$$

where

$$\mathfrak{g}_\alpha \;=\; \{\, x \in \mathfrak{g} \mid h.x \;=\; \alpha(h)\, x \text{ for all } h \in \mathfrak{h} \,\} \tag{6}$$

is the (non-trivial) weight-space associated to the functional α. Note that $\mathfrak{g}_0$ is $C_{\mathfrak{g}}(\mathfrak{h})$, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Proposition 1.2 *Assume $\mathfrak{g}$ is a semisimple Lie algebra. Then*

a) Given $\alpha, \beta \in \mathfrak{h}^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$.

b) If $x \in \mathfrak{g}_\alpha$ then $ad\, x$ is nilpotent.

c) If $\beta \neq -\alpha$ then L_α is perpendicular to L_β under the Killing form.

d) The restriction of $ad_{\mathfrak{g}}\mathfrak{h}$ to $\mathfrak{g}_0$ is non-degenerate.

Pf. Let $x_\alpha \in \mathfrak{g}_\alpha$, $x_\beta \in \mathfrak{g}_\beta$. For (a), we have

$$\begin{aligned} [h, [x_\alpha, x_\beta]] \;&=\; [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]] && (7)\\ &=\; \alpha(h)[x_\alpha, x_\beta] \;+\; \beta(h)[x_\alpha, x_\beta] \;=\; (\alpha+\beta)(h)\,[x_\alpha, x_\beta]. && (8) \end{aligned}$$

For (b), note that $(ad\, x_\alpha)^n : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+n\alpha}$. Since the number of weight spaces is finite, there is some n so $(ad\, x_\alpha)^n = 0$. For (c), by associativity of κ we have

$$0 \;=\; \kappa([h, x_\alpha], x_\beta) \;+\; \kappa(x_\alpha, [h, x_\beta]) \;=\; (\alpha(h) + \beta(h))\, \kappa(x_\alpha, x_\beta). \tag{9}$$

For (d), we know that $ad_{\mathfrak{g}}\mathfrak{h} : \mathfrak{g} \to \mathfrak{g}$ is non-degenerate, but also that $ad_{\mathfrak{g}}\mathfrak{h}\big|_{\mathfrak{g}_\alpha} = 0$ unless $\alpha = 0$. Thus $ad_{\mathfrak{g}}\mathfrak{h}$ must be non-degenerate on $\mathfrak{h}$. $\square$

Proposition 1.3 *We have $\mathfrak{g}_0 = \mathfrak{h}$.*

Pf. Write $C = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$.

Step I. *C contains the abstract semisimple and nilpotent parts of all its elements.* Since $x \in C$ has $ad\, x : \mathfrak{h} \to 0$ and $(ad\, x)_s = ad\, x_s$, $(ad\, x)_n = ad\, x_n$ are given as polynomials in $ad\, x$ without constant term, so also $(ad\, x)_s, (ad\, x)_n : \mathfrak{h} \to 0$ meaning that $x_s, x_n \in C$.

Step II. *All semisimple elements of C lie in $\mathfrak{h}$.* If x is semisimple and $x : \mathfrak{h} \to 0$ then $\mathfrak{h} + \mathbb{F}x$ is both an algebra and is toral.

Step III. *The restriction of κ to $\mathfrak{h}$ is non-degenerate.* If $x \in C$ is nilpotent, then because x commutes with $\mathfrak{h}$ we have $\kappa(x, \mathfrak{h}) = 0$. But $C = \mathfrak{h} + nilpotents$ so if $h \in \mathfrak{h}$, its dual must be in $\mathfrak{h}$.

Step IV. *C is nilpotent.* Since $C = \mathfrak{h} + nilpotents$ and C commutes with $\mathfrak{h}$, any element of C is a sum of something in $\mathfrak{h}$ with a nilpotent, and since everything commutes, and arbitrary element is nilpotent. Thus C, being ad-nilpotent, is nilpotent.

Step V. *We have $\mathfrak{h} \cap [C, C] = \{0\}$.* We have $\kappa(\mathfrak{h}, [C, C]) = \kappa([C, \mathfrak{h}], C) = 0$. Thus $[C, C]$ intersects $\mathfrak{h}$ trivially.

Step VI. *C is abelian.* If $D = [C, C]$ is non-trivial, then any $x \in D$ is nilpotent. Also $D \cap Z(C)$ is nontrivial, so we can assume $x \in D \cap Z(C)$. But then $x_n \in Z(C)$ is non-zero, so $\kappa(x_n, C) = 0$, which is impossible because κ restricted to C is non-degenerate.

Step VII. If $x \in C \setminus H$ then its nilpotent part $x_n \in C$ commutes with C, and so $\kappa(x_n, C) = 0$, which is impossible because $\kappa|_C$ is nondegenerate. $\square$

We can now identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\kappa_{\mathfrak{g}}$. Given any linear functional $\alpha \in \mathfrak{h}^*$, define $t_\alpha \in \mathfrak{h}$ by

$$\alpha(h) \;=\; \kappa(t_\alpha,\, h) \tag{10}$$

for all $h \in \mathfrak{h}$.

Proposition 1.4 *Assume $\mathfrak{g}$ is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0. Let Φ be the set of roots of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Then*

a) Φ spans $\mathfrak{h}^*$.

b) If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

c) If $\alpha \in \Phi$ and $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa(x, y)\, t_\alpha$ (recall t_α is the κ-dual of α).

d) If $\alpha \in \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$ is 1-dimensional.

e) If $\alpha \in \Phi$ then $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

f) If $\alpha \in \Phi$ and $x_\alpha \in \mathfrak{g}_\alpha$, there is some $y_\alpha \in \mathfrak{g}_{-\alpha}$ so that setting $h_\alpha = [x_\alpha, y_\alpha]$ we have $span_{\mathbb{F}}\{x_\alpha, y_\alpha, h_\alpha\} \approx \mathfrak{sl}(2, \mathbb{F})$.

g) Given $x_\alpha \in \mathfrak{g}_\alpha$, the choice of y_α in (f) leads to $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. In addition $h_\alpha = -h_{-\alpha}$.

Pf. (a). If not, there is some $h \in \mathfrak{h}$ so that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Phi$, so $h \in Z(\mathfrak{g})$, an impossibility.

(b). If $\alpha \in \Phi$ but $-\alpha \notin \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}] = \{0\}$, meaning $\mathfrak{g}_\alpha \in Z(\mathfrak{g})$, again an impossibility.

(c). Given $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, the associativity of κ implies

$$\begin{aligned} \kappa(h,\, [x,y]) &= \kappa([h,x],\, y) \\ &= \alpha(h)\, \kappa(x,\, y) \\ &= \kappa(t_\alpha,\, h)\, \kappa(x,\, y) \\ &= \kappa(t_\alpha \kappa(x,y),\, h) \end{aligned} \tag{11}$$

but then $([x,y] - \kappa(x,y)t_\alpha) \in \mathfrak{h}$ and $h \perp ([x,y] - \kappa(x,y)t_\alpha)$ for all $h \in \mathfrak{h}$, forcing $[x,y] = \kappa(x,y)t_\alpha$.

(d). Follows directly from (c).

(e). Assume $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0$. Then $\mathfrak{s} = span_{\mathbb{C}}\{x, y, t_\alpha)$ is a nilpotent Lie algebra. Consider its *ad*-representation on $\mathfrak{g}$.

(f) and (g). Given any $x_\alpha \in L_\alpha$, pick $y_\alpha \in L_{-\alpha}$ so that $\kappa(x_\alpha, y_\alpha) = \frac{2\, t_\alpha}{\kappa(t_{\alpha,t_\alpha})} \triangleq h_\alpha$. Then $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha$ and $[h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha$, so we have our copy of $\mathfrak{sl}(2, \mathbb{C})$. $\square$