1 Maximal Toral Subalgebras

A toral subalgebra of a Lie algebra $\mathfrak{g}$ is any subalgebra consisting entirely of abstractly semisimple elements.

**Lemma 1.1** If $\mathfrak{t} \subset \mathfrak{g}$ is any toral subalgebra, then $\mathfrak{t}$ is abelian.

**Pf.** Assume $x \in \mathfrak{t}$ has $(\text{ad}x)|_{\mathfrak{t}} \neq 0$. Because $x$ is semisimple, it is ad-simisimple, so there is some $y \in \mathfrak{t}$ with $(\text{ad}x)(y) = ay$. Note that

$$0 = (\text{ad}y)(\text{ad}x)(y) = -(\text{ad}y)^2x. \quad (1)$$

However, $y$ is also semisimple, so we can write

$$x = x_1 + \ldots + x_n \quad (2)$$

where $(\text{ad}y)(x_i) = \lambda_i x_i$. Therefore

$$0 = (\text{ad}y)^2x = \sum_{i=1}^{n} \lambda_i^2 x_i \quad (3)$$

so that each $\lambda_i = 0$. However this contradicts

$$0 \neq ay = (\text{ad}x)y = -(\text{ad}y)x = \sum_{i=1}^{n} \lambda_i x_i. \quad (4)$$

□

If $\mathfrak{g}$ is a semisimple Lie algebra, then any maximal toral subalgebra $\mathfrak{h}$ is called a *Cartan subalgebra* or CSA for short. Caution: in the non-semisimple case, this is not the proper use of the term CSA.
Since \( \mathfrak{h} \) is a commuting subalgebra, the action of \( \text{ad}_\mathfrak{g} \mathfrak{h} \) is simultaneously diagonalizable. This means that \( \mathfrak{g} \) has a complete root-space decomposition: there are finitely many linear functionals \( \alpha \in \mathfrak{h}^* \cong \Lambda \), \( \alpha : \mathfrak{h} \to \mathbb{F} \) so that

\[
\mathfrak{g} = \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha
\]

where

\[
\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid h.x = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}
\]

is the (non-trivial) weight-space associated to the functional \( \alpha \). Note that \( \mathfrak{g}_0 \) is \( C_\mathfrak{g}(\mathfrak{h}) \), the centralizer of \( \mathfrak{h} \) in \( \mathfrak{g} \).

**Proposition 1.2** Assume \( \mathfrak{g} \) is a semisimple Lie algebra. Then

a) Given \( \alpha, \beta \in \mathfrak{h}^* \), we have \( [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta} \).

b) If \( x \in \mathfrak{g}_\alpha \) then \( \text{ad} x \) is nilpotent.

c) If \( \beta \neq -\alpha \) then \( L_\alpha \) is perpendicular to \( L_\beta \) under the Killing form.

d) The restriction of \( \text{ad}_\mathfrak{g} \mathfrak{h} \) to \( \mathfrak{g}_0 \) is non-degenerate.

\[ \text{Pf.} \] Let \( x_\alpha \in \mathfrak{g}_\alpha \), \( x_\beta \in \mathfrak{g}_\beta \). For (a), we have

\[
[h, [x_\alpha, x_\beta]] = [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]]
\]
\[
= \alpha(h)[x_\alpha, x_\beta] + \beta(h)[x_\alpha, x_\beta] = (\alpha + \beta)(h)[x_\alpha, x_\beta].
\]

For (b), note that \( (\text{ad} x_\alpha)^n : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+n\alpha} \). Since the number of weight spaces is finite, there is some \( n \) so \( (\text{ad} x_\alpha)^n = 0 \). For (c), by associativity of \( \kappa \) we have

\[
0 = \kappa([h, x_\alpha], x_\beta) + \kappa(x_\alpha, [h, x_\beta]) = (\alpha(h) + \beta(h)) \kappa(x_\alpha, x_\beta).
\]

For (d), we know that \( \text{ad}_\mathfrak{g} \mathfrak{h} : \mathfrak{g} \to \mathfrak{g} \) is non-degenerate, but also that \( \text{ad}_\mathfrak{g} \mathfrak{h}|_{\mathfrak{g}_\alpha} = 0 \) unless \( \alpha = 0 \). Thus \( \text{ad}_\mathfrak{g} \mathfrak{h} \) must be non-degenerate on \( \mathfrak{h} \). \[ \square \]

**Proposition 1.3** We have \( \mathfrak{g}_0 = \mathfrak{h} \).

\[ \text{Pf.} \] Write \( \mathcal{C} = C_\mathfrak{g}(\mathfrak{h}) = \mathfrak{g}_0 \).

\text{Step I.} \( \mathcal{C} \) contains the abstract semisimple and nilpotent parts of all its elements. Since \( x \in \mathcal{C} \) has \( \text{ad} x : \mathfrak{h} \to 0 \) and \( (\text{ad} x)_s = \text{ad} x_s, (\text{ad} x)_n = \text{ad} x_n \) are given as polynomials in \( \text{ad} x \) without constant term, so also \( (\text{ad} x)_s, (\text{ad} x)_n : \mathfrak{h} \to 0 \) meaning that \( x_s, x_n \in \mathcal{C} \).

\text{Step II.} All semisimple elements of \( \mathcal{C} \) lie in \( \mathfrak{h} \). If \( x \) is semisimple and \( x : \mathfrak{h} \to 0 \) then \( \mathfrak{h} + \mathbb{F}x \) is both an algebra and is toral.
Step III. The restriction of \( \kappa \) to \( \mathfrak{h} \) is non-degenerate. If \( x \in C \) is nilpotent, then because \( x \) commutes with \( \mathfrak{h} \) we have \( \kappa(x, \mathfrak{h}) = 0 \). But \( C = \mathfrak{h} + \text{nilpotents} \) so if \( h \in \mathfrak{h} \), its dual must be in \( \mathfrak{h} \).

Step IV. \( C \) is nilpotent. Since \( C = \mathfrak{h} + \text{nilpotents} \) and \( C \) commutes with \( \mathfrak{h} \), any element of \( C \) is a sum of something in \( \mathfrak{h} \) with a nilpotent, and since everything commutes, and arbitrary element is nilpotent. Thus \( C \), being ad-nilpotent, is nilpotent.

Step V. We have \( \mathfrak{h} \cap [C, C] = \{0\} \). We have \( \kappa([h, [C, C]]) = \kappa([C, h], C) = 0 \). Thus \( [C, C] \) intersects \( \mathfrak{h} \) trivially.

Step VI. \( C \) is abelian. If \( D = [C, C] \) is non-trivial, then any \( x \in D \) is nilpotent. Also \( D \cap Z(C) \) is nontrivial, so we can assume \( x \in D \cap Z(C) \). But then \( x_n \in Z(C) \) is non-zero, so \( \kappa(x_n, C) = 0 \), which is impossible because \( \kappa \) restricted to \( C \) is nondegenerate.

Step VII. If \( x \in C \setminus H \) then its nilpotent part \( x_n \in C \) commutes with \( C \), and so \( \kappa(x_n, C) = 0 \), which is impossible because \( \kappa|_C \) is nondegenerate.

We can now identify \( \mathfrak{h} \) with \( \mathfrak{h}^\ast \) via \( \kappa_g \). Given any linear functional \( \alpha \in \mathfrak{h}^\ast \), define \( t_\alpha \in \mathfrak{h} \) by
\[
\alpha(h) = \kappa(t_\alpha, h)
\]
for all \( h \in \mathfrak{h} \).

**Proposition 1.4** Assume \( \mathfrak{g} \) is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0. Let \( \Phi \) be the set of roots of the adjoint action of \( \mathfrak{h} \) on \( \mathfrak{g} \). Then

a) \( \Phi \) spans \( \mathfrak{h}^\ast \).

b) If \( \alpha \in \Phi \) then \( -\alpha \in \Phi \).

c) If \( \alpha \in \Phi \) and \( x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha} \), then \( [x, y] = \kappa(x, y) t_\alpha \) (recall \( t_\alpha \) is the \( \kappa \)-dual of \( \alpha \)).

d) If \( \alpha \in \Phi \) then \( [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \leq \mathfrak{h} \) is 1-dimensional.

e) If \( \alpha \in \Phi \) then \( \alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0 \).

f) If \( \alpha \in \Phi \) and \( x_\alpha \in \mathfrak{g}_\alpha \), there is some \( y_\alpha \in \mathfrak{g}_{-\alpha} \) so that setting \( h_\alpha = [x_\alpha, y_\alpha] \) we have \( \text{span}\{x_\alpha, y_\alpha, h_\alpha\} \approx \mathfrak{sl}(2, \mathbb{F}) \).

g) Given \( x_\alpha \in \mathfrak{g}_\alpha \), the choice of \( y_\alpha \) in (f) leads to \( h_\alpha = \frac{2y_\alpha}{\kappa(t_\alpha, t_\alpha)} \). In addition \( h_\alpha = -h_{-\alpha} \).

**Pf.** (a). If not, there is some \( h \in \mathfrak{h} \) so that \( \alpha(h) = 0 \) for all \( \alpha \in \Phi \). But then \( [h, \mathfrak{g}_\alpha] = 0 \) for all \( \alpha \in \Phi \), so \( h \in Z(\mathfrak{g}) \), an impossibility.

(b). If \( \alpha \in \Phi \) but \( -\alpha \notin \Phi \) then \( [\mathfrak{g}_\alpha, \mathfrak{g}] = \{0\} \), meaning \( \mathfrak{g}_\alpha \in Z(\mathfrak{g}) \), again an impossibility.
(c). Given \( h \in \mathfrak{h} \), \( x \in \mathfrak{g}_\alpha \), \( y \in \mathfrak{g}_{-\alpha} \), the associativity of \( \kappa \) implies

\[
\kappa(h, [x, y]) = \kappa([h, x], y) \\
= \alpha(h) \kappa(x, y) \\
= \kappa(t_\alpha, h) \kappa(x, y) \\
= \kappa(t_\alpha \kappa(x, y), h)
\]

but then \( ([x, y] - \kappa(x, y)t_\alpha) \in \mathfrak{h} \) and \( h \perp ([x, y] - \kappa(x, y)t_\alpha) \) for all \( h \in \mathfrak{h} \), forcing \( [x, y] = \kappa(x, y)t_\alpha \).

(d). Follows directly from (c).

(e). Assume \( \alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = 0 \). Then \( \mathfrak{s} = \text{span}_\mathbb{C} \{x, y, t_\alpha\} \) is a nilpotent Lie algebra. Consider its \textit{ad}-representation on \( \mathfrak{g} \).

(f) and (g). Given any \( x_\alpha \in L_\alpha \), pick \( y_\alpha \in L_{-\alpha} \) so that \( \kappa(x_\alpha, y_\alpha) = \frac{2\alpha}{\kappa(t_\alpha, t_\alpha)} \triangleq h_\alpha \). Then \([h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha \) and \([h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha \), so we have our copy of \( \mathfrak{sl}(2, \mathbb{C}) \). □