Lecture 2 - Fundamental definitions, and Engel’s Theorem

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1 Basic Definitions

A representation of a Lie algebra $L$ is a homomorphism $\varphi$ of $L$ into the Lie algebra $\mathfrak{gl}(V)$ for some vector space $V$ over $F$. Every Lie algebra has at least one representation, the adjoint representation $ad : L \to End(V)$.

A subalgebra $K$ of $L$ is a subspace that is closed under the bracket.

An ideal $I$ of $K$ is a subalgebra so that $x \in L$, $y \in I$ implies $[x,y] \in I$.

If $I$ and $J$ are ideals, so is $I + J$, defined to be the set of all elements $c_1x + c_2y$ where $c_1, c_2 \in F$, $x \in I$, $y \in J$.

If $I$ and $J$ are ideals, so is $[I,J]$, which is defined to be the vector space spanned by elements of the form $[x,y]$ where $x \in J$ and $y \in J$.

Two ideals possessed by any Lie algebra are the derived algebra $[L,L]$, and the center $Z(L)$ or $C(L)$, defined to be the set of elements $x \in L$ so that $[x,y] = 0$ for all $y \in L$. Either of these algebras may be trivial or may equal $L$ itself.

**Proposition 1.1 (Humphreys 2.2)** Assuming $I, J \subset L$ are ideals. Then $(I + J)/J$ is canonically isomorphic to $I/(I \cap J)$, and if $I \subset J$ then $K/J$ is canonically isomorphic to $(K/I)/(J/I)$

If $K \subseteq L$ is a subalgebra, we define the normalizer of $K$ in $L$

$$N_L(K) = \{ x \in L \mid [x,K] \in K \}.$$  \hfill (1)

Using the Jacobi identity, $N_L(K)$ can be seen to be a subalgebra of $L$. It is the largest subsalgebra that contains $K$ as an ideal. If $N_L(K) = L$ then $K$ is an ideal. If $N_L(K) = K$ then $K$ is said to be self-normalizing.
If $K \subseteq L$ is a subalgebra, we define the \textit{centralizer} of $K$ in $L$ to be

$$C_L(K) = \{ x \in L \mid [x,K] = 0 \}.$$  \hfill (2)

Note that $K$ is usually not contained in $C_L(K)$. Also, $C_L(L) = C(L)$.

A \textit{derivation} of a Lie algebra $L$ is a linear map $\delta : L \to L$ so that $\delta [x,y] = [\delta x, y] + [x, \delta y]$. The vector space of derivations $\text{Der}(L)$ is in fact a Lie algebra; it is easily checked that if $\delta, \delta' \in \text{Der}(L)$ then $[\delta, \delta'] \triangleq \delta \delta' - \delta' \delta$ is also a derivation. By the Jacobi identity, the adjoint map can be thought of as $ad : L \to \text{Der}(L)$.

If $[L,L] = \{0\}$ then $L$ is called \textit{abelian}. If $L$ has no nontrivial proper ideals, then $L$ is called \textit{simple}.

If $L, N$ are Lie algebras, a \textit{homomorphism} $\varphi : L \to N$ is a linear map that commutes with the brackets; that is

$$\varphi([x,y]) = [\varphi(x), \varphi(y)]$$  \hfill (3)

whenever $x, y \in L$. The \textit{Kernel} of $\varphi$, denote $\text{Ker}(\varphi)$ is the vector space kernel of the map $\varphi$. It is easily checked that $\text{Ker}(\varphi)$ is an ideal.

A homomorphism $\varphi : L \to N$ is called a \textit{monomorphism} if $\text{Ker}(\varphi)$ is the trivial subspace. It is called an epimorphism if its image is $N$. It is called an isomorphism if it is a monomorphism and an epimorphism.

\textbf{Proposition 1.2 (Humphreys 2.2)} If $\varphi : L \to N$ is a homomorphism, then $\text{Im}(\varphi)$ is canonically isomorphic to $L/\text{Ker}(\varphi)$.

A Lie algebra $L$ is called a linear Lie algebra if it is a subalgebra of $\text{gl}(V)$ for some finite dimensional vector space $V$.

\textbf{Proposition 1.3} Any simple Lie algebra is isomorphic to a linear Lie algebra

\section{Solvable Lie algebras}

Let $L$ be a Lie algebra. We can define its derived series $L^{(0)}, L^{(1)}, \ldots$ by $L^{(0)} = L$ and

$$L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right].$$  \hfill (4)

We call $L$ \textit{solvable} if $L^{(n)} = \{0\}$ for some $n$.

\textbf{Proposition 2.1 (Humphreys 3.1)} Let $L$ be a Lie algebra.
a) If $L$ is solvable, so are all subalgebras and all homomorphic images.

b) If $I \subseteq L$ is a solvable ideal and $L/I$ is solvable, then $L$ is solvable.

c) If $I, J \subseteq L$ are solvable ideals, then $I + J$ is a solvable ideal.

Pf. Easy.

3 Nilpotent Lie algebras

Let $L$ be a Lie algebra. We can define its descending central series $L^0, L^1, \ldots$ by $L^0 = L$ and

$$L^k = [L, L^{k-1}] .$$

(5)

We call $L$ nilpotent if $L^n = \{0\}$ for some $n$.

**Proposition 3.1 (Humphreys 3.2)** Let $L$ be a Lie algebra.

a) If $L$ is nilpotent, so are all subalgebras and all homomorphic images.

b) If $L/Z(L)$ is nilpotent, so is $L$.

c) If $L$ is nilpotent, then $Z(L)$ is not trivial.

Pf. a) Easy

b) If $L/Z(L)$ is nilpotent then $L^n \subseteq Z(L)$ for some $L$. Then $L^{n+1} = \{0\}$.

c) There is some $n$ so that $L^n = \{0\}$ but $L^{n-1} \neq \{0\}$. Clearly $\{0\} = L^n = [L, L^{n-1}]$ implies $L^{n-1} \subseteq Z(L)$.

**Definition** An element $x \in L$ is called **ad-nilpotent** if $(adx)^n = 0$ for some $n$. A Lie algebra $L$ is called **ad-nilpotent** if every element of $L$ is ad-nilpotent.

If $x, y \in L$, then $(adx)^n y = [x, [x, \ldots, [x, y] \ldots]] \in L^{n+1}$. Thus if $L$ is nilpotent, it is ad-nilpotent.

**Theorem 3.2 (Engel’s Theorem)** If $L$ is ad-nilpotent, it is nilpotent

**Theorem 3.3** If $L$ is a subalgebra of $\mathfrak{gl}(V)$ ($V$ finite dimensional) and every $x \in L$ is a nilpotent transformation (meaning given $x$ there is some $n \in \mathbb{N}$ so that $x^n.v = 0$ whenever $v \in V$), then there is some $v \in V$ so that $x.v = 0$ for all $x \in L$. 
Induction on the dimension of \( L \). The theorem is clearly true for all \( L \) with \( \dim(L) = 1 \). This is because \( x \in L \) implies \( x^n.v = 0 \) for some \( n \), so that there is a largest \( i \in \mathfrak{K} \) with \( x^i.v \neq 0 \) but \( x^{i+1}.v = 0 \), in which case \( v_i = x^i.v \) is a zero eigenvector.

Assume the theorem is true for all Lie algebras \( K \) with \( \dim(K) < \dim(L) \). Let \( K \) be any maximal subalgebra of \( L \)—clearly subalgebras exist, for instance 1-dimensional subalgebras. We will prove first that \( K \) has codimension 1, and since the theorem is true for the action of \( K \) on \( V \) we are left just a single basis element whose action must be checked.

To prove \( K \) has codimension 1, consider the adjoint action of \( K \) on \( L/K \) (of course \( L/K \) is not a Lie algebra but only a vector space; still the action of \( K \) is well-defined (check)). By the inductive hypothesis, there is some vector \( z \in L \) so that \( z + K \in L/K \) is a zero-eigenvector for every element of \( K \). This means that \( K + \mathbb{F}z \) is also a subalgebra that strictly contains \( K \), implying that either \( K \) was not maximal (which it is) or that \( K + \mathbb{F}z \) is in fact \( L \), verifying that \( K \) has codimension 1. Since \( L = K + \mathbb{F}z \) and we showed \([z, k] \in K\) when \( k \in K \), \( K \) is an ideal.

Now let \( W \subseteq V \) be the subspace consisting of all zero-eigenvectors for \( K \), or

\[
W = \{ w \in V \mid k.w = 0 \text{ when } k \in K \}.
\] (6)

To see that \( L \) fixes \( W \), let \( w \in W \), \( y \in L \), and \( k \in K \). Since

\[
k.y.w = y.k.w + [k, y].w
\] (7)

Because \( k, [k, y] \in K \), the right-side is zero. Therefore \( y.w \in W \). Using again \( L = K + \mathbb{F}z \) and knowing that \( z \) has a nilpotent action on \( V \) and therefore on \( W \), there must be a zero-eigenvector \( w' \) for \( z \) in \( W \). Thus \( w' \) is a zero-eigenvector for all of \( L \). □

Proof of Engel’s theorem. Again we argue inductively on the dimension of \( L \), assuming Engel’s theorem holds for all Lie algebras \( K \) with \( \dim(K) < \dim(L) \). The adjoint action \( \text{ad} : L \to \mathfrak{gl}(L) \) expresses \( L \) as an algebra of nilpotent endomorphisms, which therefore have a common zero-eigenvector. Thus \( Z(L) \) is non-trivial. Since \( L/Z(L) \) is ad-nilpotent, and therefore nilpotent by the induction hypothesis, \( L \) is also nilpotent. □