Lecture 4 - The Fitting and Jordan-Chevalley decompositions

September 18, 2012

1 Minimal and Characteristic polynomials

If $x \in End(V)$, its *minimal polynomial* M is the monic polynomial of smallest degree so that the transformation $M(x) \in End(V)$ is zero. The *characteristic polynomial* Ch is a polynomial in one variable given by

$$Ch(\lambda) \;=\; \det(\lambda I \,-\, x). \tag{1}$$

Clearly it is monic. Plugging x itself into the resulting polynomial and formally evaluating, we clearly have that $Ch(x)$ is the zero endomorphism. We have

$$\begin{aligned} M(\lambda) \;&=\; \prod_{i=1}^{n} (\lambda \,-\, \lambda_i)^{e_i} && (2) \\ Ch(\lambda) \;&=\; \prod_{i=1}^{n} (\lambda \,-\, \lambda_i)^{f_i} && (3) \\ & && (4) \end{aligned}$$

and that $f_i \geq e_i$. The numbers λ_i are the eigenvalues of x, f_i is the multiplicity of λ_i, and e_i is the degree of λ_i.

2 The Fitting Decomposition

Let x be a linear operator on a finite dimensional vector space V. Then V is isomophic to a direct sum $V = V_{0,x} \oplus V_{1,x}$, where $V_{i,x}$ is a subspace of V. Further, $V_{0,x}$ has the property that $x.V_{0,x} \subset V_{0,x}$ and for some $n \in \mathbb{N}$ we have

$$x^n.V_{0,x} \;=\; \{0\}, \tag{5}$$

and $V_{1,x}$ has the property that

$$x.V_{1,x} \;=\; V_{1,x} \tag{6}$$

or that x is an isomorphism on $V_{1,x}$. With respect to x, $V_{0,x}$ is called the *Fitting null space* and $V_{1,x}$ is called the *Fitting one space.*

First, define

$$V_{0,x} \;=\; \{\, v \in V \mid x^i.v \text{ for some } i \in \mathbb{N}\,\}. \tag{7}$$

Since

$$V \;\supseteq\; x.V \;\supseteq\; x^2.V \;\supseteq\; \dots \tag{8}$$

and V is finite dimensional, there is some r so that $x^r.V = x^{r+1}.V = \dots$. Simply define $V_{1,x}$ to be this $x^r.V$.

Clearly $V_{0,x}$ and $V_{1,x}$ are invariant under x. Now let $w \in V$; we show that $w \in V_{0,x} \oplus V_{1,x}$. Letting $r \in \mathbb{N}$ be large enough that $x^r.V_{0,x} = \{0\}$, we see that $x^r.V = V_{1,x}$ implies

$$x^r.w \;\in\; V_{1,x}. \tag{9}$$

But x, hence x^r is an isomorphism on $V_{1,x}$, so there is some $w_1 \in V_{1,x}$ with $x^r.w = x^r.w_1$. But then $x^r.(w - w_1) = 0$, so setting $w_0 = w - w_1$ we have

$$w \;=\; w_0 \,+\, w_1. \tag{10}$$

A further decomposition exists. If μ is any polynomial in one variable, set

$$V_\mu \;=\; \{\, v \in V \mid \mu(x)^r.V = 0 \text{ for some } r \in \mathbb{N}\,\}. \tag{11}$$

Let π_i be the irreducible factors of the minimal polynomial of x, so $M(\lambda) = \prod_{i=1}^n \pi_i(\lambda)^{e_i}$, and consider the spaces V_{π_i}. Since x commutes with $\pi_i(x)$, we have $x.V_{\pi_i} \subseteq V_{\pi_i}$.

Proposition 2.1 *If $\pi_i(\lambda) \neq \lambda$, then the restriction of x to V_{π_i} is an isomorphism. Futher, it is the sum of two operators $x_{s,i}$ and $x_{n,i}$, with the following properties. The operator $x_{s,i}$ acts on V_{π_i} by constant multiplication, and $x_{n,i} : V_{\pi_i} \to V_{\pi_i}$ is a nilpotent endomorphism. Finally, $x_{s,i}$ and $x_{n,i}$ are the unique operators on V_{π_i} with these properties.*

Pf. Were $x : V_{\pi_i} \to V_{\pi_i}$ to have a zero eigenvector v_0, then

$$\begin{aligned} 0 \;=\; \pi_i(x)^{e_i}.v_0 \;&=\; (x \,-\, \lambda_i\, I)^{e_i - 1} \,.\, (x \,-\, \lambda_i\, I)\,.v_0 && (12)\\ &=\; \lambda_i\, (x \,-\, \lambda_i\, I)^{e_i - 1}\,.v_0 && (13)\\ &\;\;\vdots && (14)\\ &=\; \lambda_i^{e_i} v_0 && (15) \end{aligned}$$

so $\lambda_i = 0$, which is false by hypothesis. Thus $x : V_{\pi_i} \to V_{\pi_i}$ is an isomorphism.

Of course x is the sum of the operators $x_{s,i} = \lambda_i I$ and $x_{n,i} = x - \lambda_i I$, and $x - \lambda_i I$ is nilpotent on V_{π_i} by definition. The three operators x, $\lambda_i I$, and $x - \lambda_i I$ clearly commute.

If $x'_{s,i}$ is another operator on V_{π_i} that acts by constant multiplication and $x - x'_{s,i}$ is nilpotent, then $x'_{s,i} - x_{s,i}$ acts by constant multiplication. Clearly also $x'_{s,i}$ and $x_{s,i}$ commute; an easy computation then shows $x'_{n,i}$ and $x_{n,i}$ commute as well. Also, $x'_{n,i} - x_{n,i} = (x - x'_{s,i}) - (x - x_{s,i}) = x_{s,i} - x'_{s,i}$. Since $x'_{n,i} - x_{n,i}$ is nilpotent, so is $x_{s,i} - x'_{s,i}$, which is impossible unless it is zero. □

Proposition 2.2 *Let $M(\lambda) = \prod_{i=0}^n \pi_i(\lambda)^{e_i}$ be the minimal polynomial for the endomorphism $x : V \to V$, where $\pi_0(\lambda) = \lambda$ (and possibly $e_0 = 0$). The vector space $V_{1,x}$ has the decomposition*

$$V_{1,x} \;=\; \bigoplus_{i=1}^{n} V_{\pi_i} \tag{16}$$

Therefore the vector space V has the decomposition

$$V \;=\; \bigoplus_{i=0}^{n} V_{\pi_i}. \tag{17}$$

Pf. We have $V_{0,x} = V_{\pi_0}$ by definition. Since x is an isomorphism on each V_{π_i} when $i \neq 0$, we have

$$x^r.\left(\bigoplus_{i=1}^{n} V_{\pi_i}\right) \;=\; \bigoplus_{i=1}^{n} V_{\pi_i} \tag{18}$$

so that $\bigoplus_{i=1}^n V_{\pi_i} \subseteq V_{1,x}$. If there is some $v \in V_{1,x} \setminus \bigoplus_{i=1}^n V_{\pi_i}$, then

$$W \;=\; V_{1,x}/\bigoplus_{i=1}^{n} V_{\pi_i} \tag{19}$$

is non-trivial, and is acted on by x. Further, because the action of x on $V_{1,x}$ is by isomorphism, its action on W is also an isomorphism. Therefore there is some $w \in W$ with $x.w = cw$ for some non-zero $c \in \mathbb{F}$. Letting $v' \in V_{1,x}$ be in the inverse image of w under the quotient $V_{1,x} \to W$, we have

$$x.v' \;=\; cv' + v_1 \tag{20}$$
$$x^{e_0}.v' \;=\; c^{e_0}v' + v_2 \tag{21}$$

where $v_1, v_2 \in \bigoplus_{i=1}^n V_{\pi_i}$ Now let $Q(\lambda) = \prod_{i=1}^n \pi_i(\lambda)^{e_i}$. We have that $Q(x)$ kills $\bigoplus_{i=1}^n V_{\pi_i}$, $x^{e_0}Q(x)$ kills V, and $Q(x)$ commutes with x. Thus

$$x^{e_0}.v' \;=\; c^{e_0}v' + v_2 \tag{22}$$
$$x^{e_0}.Q(x).v' \;=\; c^{e_0}Q(x).v'. \tag{23}$$

Since c is non-zero, this forces $Q(x)$ to kill v', which means $v' \in \bigoplus_{i=1}^{n} V_{\pi_i}$.

$\square$

Corollary 2.3 (The Jordan-Chevalley decomposition) *If x is an endomorphism on V, then $x = x_s + x_n$, where x_s and x_n have the following properties: the roots of the minimal polynomial of x_s are distinct, x_n is nilpotent, and x_s and x_n commute. Further, x_s and x_n are the unique operators that satisfy these properties.*

Pf. By Proposition (2.1), we have $x|_{V_{\pi_i}} = x_{s,i} + x_{n,i}$, where all three operators commute. We can extend the action of $x_{s,i}, x_{n,i}$ from V_{π_i} to V by requiring them to act as multiplication by zero on all V_{π_j} when $i \neq j$. This makes sense because $V = \bigoplus_{i=0}^{n} V_{\pi_i}$, so we have well-defined projections onto the V_{π_i}. Further, all the $x_{s,i}, x_{n,i}$ operators commute. Defining

$$x_s \;=\; \sum_{i=0}^{n} x_{s,i} \qquad \text{and} \qquad x_n \;=\; \sum_{i=0}^{n} x_{n,i} \tag{24}$$

provides the required operators. Now assume x'_s and x'_n are other choices for the semi-simple and nilpotent parts of x. But then $x_s - x'_s = x_n - x'_n$. Because $x - x'_s$ is nilpotent, x and x'_s have the same eigenspace decomposition, so x'_s acts by constant multiplication on each V_{π_i}. Clearly then $x_s = x'_s$, so also $x_n = x'_n$. $\square$

If $x : V \to V$ is an endomorphism, x_s is called its *semisimple* part and x_n is called its *nilpotent* part.

Proposition 2.4 *Given an endomorphism $x : V \to V$, where V is a finite dimensional vector space over the algebraically complete field $\mathbb{F}$, there are polynomials $P_s(\lambda)$, $P_n(\lambda)$ without constant terms so that $x_s = P_s(x)$ and $x_n = P_n(x)$. Further, x, x_s, and x_n commute, and if x'_s, x'_n are other semi-simple and nilpotent endomorphisms with $x = x'_s + x'_n$, then $x'_s = x_s$ and $x'_n = x_n$.*

Pf. As usual let

$$M(\lambda) \;=\; \prod_{i=0}^{n} (\lambda \,-\, \lambda_i)^{e_i} \tag{25}$$

be the minimal polynomial of x. Consider the (commutative) algebra in $End(V)$ generated by x and the identity I. By the Chinese remainder theorem, we can find an element P_s of this algebra that satisfies the following congruences:

$$P_s(x) \;\equiv\; 0 \quad (mod\; x) \tag{26}$$

$$P_s(x) \;\equiv\; \lambda_1 I \quad (mod\; (x \,-\, \lambda_1 I)^{e_1}\;) \tag{27}$$

$$\vdots \tag{28}$$

$$P_s(x) \;\equiv\; \lambda_n I \quad (mod\; (x \,-\, \lambda_n I)^{e_n}\;) \tag{29}$$

Set $P_n(x) = x - P_s(x)$. Since $P_s(x) \equiv 0\,(mod\,x)$ we have that x is a factor of $P_s(x)$, so neither P_s nor P_n have a constant term.

These congruences mean in particular that $P_s(x) - \lambda_i I = q_i(x) \cdot (x - \lambda_i I)^{e_i}$ for some polynomials $q_i(x)$. Since $(x - \lambda_i I)^{e_i}$ acts by 0 on V_{π_i}, we have that $P_s(x)$ acts as constant multiplication by λ_i on V_{π_i}. Thus $P_s(x)$ is semisimple. Define $x_s = P_s(x)$ and $x_n = P_n(x)$. Clearly x_s commutes with x. Further, it is clear that the only eigenvalue of $x_n = x - x_s$ is zero, so x_n is nilpotent.

If x_s' were another semisimple endomorphism with $x - x_s'$ nilpotent, then since $x - x_s'$ has all zero eigenvalues, it must be the case that x_s' and x_s have the same action on each V_{π_i}. Thus they are the same. $\square$

It should be noted that x acts on L via the adjoint representation. Sometimes this allows x to be separated into its *abstract* semisimple and nilpotent parts. Clearly $ad\,x \in End(L)$ has components $ad\,x = (ad\,x)_s + (ad\,x)_n$, but when does $(ad\,x)_s = ad\,x_s$, $(ad\,x)_n = ad\,x_n$ for elements $x_s, x_n \in L$? The most we will prove today is the following.

Proposition 2.5 *Assume $x \in End(V)$ has Jordan decomposition $x = x_s + x_n$. Then $ad\,x \in End(End(V))$ has Jordan decomposition $ad\,x = ad\,x_s + ad\,x_n$.*

Pf. An inductive argument easily shows that $ad\,x_n$ is nilpotent if x_n is; indeed if $N \in \mathbb{N}$ has $(x_n)^N = 0$ then $(ad\,x_n)^{2N} = 0$. To see that $ad\,x_s$ is semi-simple when x_s is, pick a basis $\{v_1, \ldots, v_n\}$ so that $x_s.v_i = \lambda_i\, v_i$. Then $V = span\{v_1, \ldots, v_n\}$ is identified with $\mathbb{C}^n$, so we can let the usual matrices $\{e_{ij}\}_{i,j=1}^n$ be the basis for $\mathfrak{gl}(V)$. We have $e_{ij}(v_k) = \delta_{jk} v_i$, and

$$\begin{aligned}(ad\,x_s\,(e_{ij})\,)v_k \;&=\; x_s.e_{ij}.v_k \;-\; e_{ij}.x_s.v_k \\ &=\; \delta_{jk} x_s.v_i \;-\; \lambda_k\, e_{ij}.v_k \\ &=\; (\lambda_i - \lambda_k)\;\delta_{jk}\,v_i \\ &=\; (\lambda_i - \lambda_j)\;\delta_{jk}\,v_i \;=\; (\lambda_i - \lambda_j)\;e_{ij}.v_k\end{aligned} \tag{30}$$

so $(ad\,x_s)(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$. Thus $ad\,x_s$ acts diagonally on $\mathfrak{gl}(V)$ and is therefore semisimple. By the uniqueness of the Jordan decomposition therefore $(ad\,x)_s = ad\,x_s$ and $(ad\,x)_n = ad\,x_n$. $\square$