1 New modules from old

A few preliminaries are necessary before jumping into the representation theory of semisimple algebras. First a word on creating new \( g \)-modules from old. Any Lie algebra \( g \) has an action on a 1-dimensional vector space (or \( \mathbb{F} \) itself), given by the trivial action. Second, any action on spaces \( V \) and \( W \) can be extended to an action on \( V \otimes W \) by forcing the Leibnitz rule: for any basis vector \( v \otimes w \in V \otimes W \) we define

\[
x.(v \otimes w) = x.v \otimes w + v \otimes x.w
\]

One easily checks that \( x.y.(v \otimes w) - y.x.(v \otimes w) = [x,y].(v \otimes w) \). Assuming \( g \) has an action on \( V \), it has an action on its dual \( V^* \) (recall \( V^* \) is the vector space of linear functionals \( V \to \mathbb{F} \)), given by

\[
(v.f)(x) = -f(x.v)
\]

for any functional \( f : V \to \mathbb{F} \) in \( V^* \). This is in fact a version of the “forcing the Leibnitz rule.” That is, recalling that we defined \( x.(f(v)) = 0 \), we define \( x.f \in V^* \) implicitly by

\[
x.(f(v)) = (x.f)(v) + f(x.v).
\]

For any vector spaces \( V, W \), we have an isomorphism

\[
\text{Hom}(V, W) \approx V^* \otimes W;
\]

so \( \text{Hom}(V, W) \) is a \( g \)-module whenever \( V \) and \( W \) are. This can be defined using the above rules for duals and tensor products, or, equivalently, by again forcing the Leibnitz rule: for \( F \in \text{Hom}(V, W) \), we define \( x.F \in \text{Hom}(V, W) \) implicitly by

\[
x.(F(v)) = (x.F)(v) + F(x.v).
\]
2 Schur’s lemma and Casimir elements

**Theorem 2.1 (Schur’s Lemma)** If \( g \) has an irreducible representation on \( \text{gl}(V) \) and if \( f \in \text{End}(V) \) commutes with every \( x \in g \), then \( f \) is multiplication by a constant.

*Pf.* The operator \( f \) has a complete eigenspace decomposition, which is preserved by every \( x \in g \). Namely if \( v \in V \) belongs to the generalized eigenspace with eigenvector \( \lambda \), meaning \((f - \lambda I)^k.v = 0 \) for some \( k \), then
\[
(f - \lambda I)^k.x.v = x.(f - \lambda I)^k.v = 0. \tag{6}
\]
Thus the generalized \( \lambda \)-eigenspace is preserved by \( g \) and is therefore a sub-representation. By irreducibility, this must be all of \( V \). Clearly then \( f - \lambda I \) is a nilpotent operator on \( V \) that commutes with \( g \). Thus \( V_0 = \{ v \in V \mid (f - \lambda I).v = 0 \} \) is non-trivial. But \( V_0 \) is preserved by \( g \), so must equal \( V \). Therefore \( f = \lambda I \). \( \square \)

Now assume \( V \) is a \( g \)-module, or specifically that a homomorphism \( \varphi : g \to \text{gl}(V) \) exists. As with the adjoint representation we can establish a bilinear form \( B_{\varphi} : g \times g \to \mathbb{F} \)
\[
B_{\varphi}(x, y) = \text{Tr} (\varphi(x) \varphi(y)). \tag{7}
\]
If \( \varphi \) is the adjoint map, of course this is the Killing form. Clearly
\[
B_{\varphi}([x, y], z) = B_{\varphi}(x, [y, z]) \tag{8}
\]
so that the radical of \( B_{\varphi} \) is an ideal of \( g \). Also, the Cartan criterion implies that the image under \( \varphi \) of the radical of \( B_{\varphi} \) is solvable.

Thus if \( \varphi \) is a faithful representation of a semisimple algebra, \( B_{\varphi} \) is non-degenerate. Letting \( \{ x_i \}_{i=1}^n \) be a basis for \( g \), a (unique) dual basis \( \{ y_i \}_{i=1}^n \) exists, meaning the \( y_i \) satisfy
\[
B_{\varphi}(x_i, y_j) = \delta_{ij}. \tag{9}
\]
We define the *casimir element* \( c_{\varphi} \) of such a representation by
\[
c_{\varphi} = \sum_{i=1}^n \varphi(x_i)\varphi(y_i) \in \text{End} V. \tag{10}
\]

**Lemma 2.2** Given a faithful representation \( \varphi \) of a semisimple Lie algebra, the casimir element commutes with all endomorphisms in \( \varphi(g) \).

*Pf.* Let \( x \in g \) be arbitrary, and define constants
\[
[x, x_i] = a_{ij} x_j \quad [x, y_i] = b_{ij} x_j \tag{11}
\]
We have

\[- b_{ji} = - \sum_{k=1}^{n} b_{jk} \delta_{ik} = - B_{\varphi}(x_{i}, [x, y_{j}]) = B_{\varphi}([x, x_{i}], y_{k}) = \sum_{k=1}^{n} a_{ij} \delta_{jk} = a_{ij}\]  

(12)

Therefore

\[\begin{align*}
[\varphi(x), c_{\varphi}] &= \sum_{i=1}^{n} [\varphi(x), \varphi(x_{i})\varphi(y_{i})] \\
&= \sum_{i=1}^{n} [\varphi(x), \varphi(x_{i})] \varphi(y_{i}) + \sum_{i=1}^{n} \varphi(x_{i}) [\varphi(x), \varphi(y_{i})] \\
&= \sum_{i=1}^{n} \varphi([x, x_{i}]) \varphi(y_{i}) + \sum_{i=1}^{n} \varphi(x_{i}) \varphi([x, y_{i}]) \\
&= \sum_{i,j=1}^{n} a_{ij} \varphi(x_{j}) \varphi(y_{i}) + \sum_{i,j=1}^{n} b_{ij} \varphi(x_{i}) \varphi(y_{j}) \\
&= 0
\end{align*}\]  

(13)

Lemma 2.3 If \( \varphi : g \rightarrow gl(V) \) is an irreducible, faithful representation of the semisimple Lie algebra \( g \), then the Casimir endomorphism \( c_{\varphi} \) acts by constant multiplication, with the constant equal to \( \dim(g)/\dim(V) \).

Pf. That \( c_{\varphi} \) acts by constant multiplication by some \( \lambda \in \mathbb{F} \) follows from Schur’s lemma. We see that

\[\begin{align*}
\text{Tr}(c_{\varphi}) &= \sum_{i=1}^{\dim(g)} \text{Tr}(\varphi(x_{i})\varphi(y_{i})) = \sum_{i=1}^{\dim(g)} B_{\varphi}(x_{i}, y_{i}) = \dim(g)
\end{align*}\]  

(14)

and also that \( \text{Tr}(c_{\varphi}) = \lambda \cdot \dim(V) \). Thus \( \lambda = \dim(g)/\dim(V) \).

\[\square\]

3 Weyl’s Theorem

Lemma 3.1 If \( \varphi : g \rightarrow gl(V) \) is a representation and \( g \) is semisimple, then \( \varphi(g) \subseteq gl(V) \).

Pf. Because \( [g, g] = g \), we have \( [\varphi(g), \varphi(g)] = \varphi([g, g]) = \varphi(g) \).

\[\square\]

Theorem 3.2 (Weyl) Let \( \varphi : g \rightarrow gl(V) \) be a representation\(^1\) of a semisimple Lie algebra. Then \( \varphi \) is completely reducible.

\(^1\)under the usual conditions: \( g \) and \( V \) are finite dimensional, and the field is algebraically closed and of characteristic 0.
Thus we have that \( F \) operator acts as an element of \( g \) and since \( \varphi \) has codimension 1 submodule of \( V/W \), by multiplication by 0. All this means that \( c_\varphi : V \to V \) has a 1-dimensional Kernel that trivially intersects \( W \), so

\[
V = W \oplus \text{Ker}(c_\varphi).
\]  
(15)

Since \( c_\varphi \) commutes with \( \varphi(g) \), we have that \( \text{Ker}(c_\varphi) \) is indeed a (trivial) \( g \)-module.

Step II: Case of a general codimension 1 irreducible submodule. Let \( W \subset V \) be an arbitrary codimension 1 submodule of \( g \). If \( W \) is not irreducible, there is another submodule \( V_1 \subset W \), which we can assume to be maximal. Then \( V/W_1 \) is an irreducible submodule of \( V/W \), and still has codimension 1. Thus by step I, we have

\[
V/W_1 = W/W_1 \oplus V_1/W_1,
\]  
(16)

where \( V_1/W_1 \) is a 1-dimensional submodule of \( V/W_1 \). Because \( \dim(W) \neq 0 \), we have \( \dim(V_1) < \dim(V) \). We also have that \( W_1 \) is a codimension 1 submodule of \( V_1 \).

Since \( \dim(V_1) < \dim(V) \), an induction argument lets us assert \( V_1 \) that \( V_1 = W_1 \oplus \mathbb{F}z \), for some \( z \in V_1 \), as \( g \)-modules. Note that \( \mathbb{F}z \cap W = \{0\} \), so \( V = W \oplus \mathbb{F}z \) as vector spaces; the question is whether this is a \( g \)-module decomposition. However because \( V/W_1 = (W/W_1) \oplus (V_1/W_1) \), we have \( g.W \subseteq W \), so indeed \( W \oplus \mathbb{F}z \) is a \( g \)-module decomposition.

Step III: The general case. Assume \( W \subset V \) is submodule of strictly smaller dimension, and let \( \mathcal{V} \subset \text{Hom}(V, W) \) be the subspace of \( \text{Hom}(V, W) \) consisting of maps that act by constant multiplication on \( W \). Let \( W \subset \mathcal{V} \) be the subset of maps that act as multiplication by zero on \( W \). Moreover, \( \mathcal{V} \subset \mathcal{W} \) has codimension, as any element of \( \mathcal{V}/W \) is determined by its scalar action on \( W \).

However we can prove that \( \mathcal{V} \) and \( W \) are \( g \)-modules. Letting \( F \in \mathcal{V}, w \in W \), and \( x \in g \), we have that \( F(w) = \lambda w \) for some \( \lambda \in \mathbb{F} \) and, since \( x.w \in W \) also \( F(x.w) = \lambda x.w \). Thus

\[
(x.F)(w) = x.(F(w)) - F(x.w) = x.(\lambda w) - \lambda(x.w) = 0.
\]  
(17)

Thus all operators in \( g \) take \( \mathcal{V} \) to \( W \), so in particular they are both \( g \)-modules.

By Step II above, there is a \( g \)-submodule in \( \mathcal{V} \) complimentary to \( W \), spanned by some operator \( F_1 \). Scaling \( F_1 \) we can assume \( F_1|_W \) is multiplication by 1. Because \( F_1 \) generates a 1-dimensional submodules and \( g \) acts as an element of \( \mathfrak{sl}(1, \mathbb{C}) \approx \{0\} \), we have \( g.F_1 = 0 \). Thus we have that \( x \in g, v \in V \) implies

\[
0 = (x.F_1)(v) = x.(F_1(v)) - F_1(x.v).
\]  
(18)
This is the same as saying $F_1$ is a $\mathfrak{g}$-module homomorphism $V \to W$. Its kernel is therefore a $\mathfrak{g}$ module, and, since $F_1$ is the identity on $V$ and maps $V$ to $W,$ must be complimentary as a vector space to $W$. Therefore

$$V = W \oplus Ker(F_1) \quad \text{(19)}$$

as $\mathfrak{g}$-modules.

□