has a Lyapunov function given by $L(x, y) = x^2 + y^2$. To verify this assertion, we compute the derivative of L along a solution $(x(t), y(t))$ by

$$\begin{aligned}\frac{d}{dt}L(x(t), y(t)) &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ &= 2x(-x+y) + 2y(-x-y) \\ &= -2(x^2+y^2) \\ &\leq 0.\end{aligned}$$

Since $dL/dt = 0$ only at the origin, this function decreases along all nonequilibrium solution curves. But L cannot be a gradient system, since the eigenvalues at (0,0) are complex $(-1 \pm i)$; so the origin is a spiral sink, which cannot occur in gradient systems.

EXERCISES FOR SECTION 5.4

1. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -x^3 \\ \frac{dy}{dt} &= -y^3.\end{aligned}$$

(a) Verify that $L(x, y) = (x^2 + y^2)/2$ is a Lyapunov function for the system.

(b) Sketch the level sets of L.

(c) What can you conclude about the phase portrait of the system from the information given in parts (a) and (b) above? (Sketch the phase portrait and write a short essay describing what you know about the phase portrait and how you know it.)

2. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - \frac{y}{4} + x^2.\end{aligned}$$

(a) Verify that the function

$$L(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{3}$$

is a Lyapunov function for the system.

(b) Sketch the level sets of L.

(c) What can you conclude about the phase portrait of the system from the information in parts (a) and (b)? (Sketch the phase portrait and write a short essay describing what you know about the phase portrait and how you know it.)

3. Consider the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -4x - 0.1y.$$

(a) Verify that all solutions tend toward the origin as t increases, and sketch the phase portrait. [*Hint*: The system is linear.]

(b) Verify that $L(x, y) = x^2 + y^2$ is *not* a Lyapunov function for the system.

(c) Verify that $K(x, y) = 2x^2 + y^2/2$ is a Lyapunov function for the system.

In Exercises 4–11, we consider the damped pendulum system

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = -\frac{g}{l}\sin\theta - \frac{b}{m}v,$$

where b is the damping coefficient, m is the mass of the bob, l is the length of the arm, and g is the acceleration of gravity ($g \approx 9.8$ m/s^2).

4. What relationship must hold between the parameters b, m, and l for the period of a small swing back and forth of the damped pendulum to be one second?

5. Suppose we have a pendulum clock that uses a slightly damped pendulum to keep time (that is, b is positive but $b \approx 0$). The clock "ticks" each time the pendulum arm crosses $\theta = 0$. If the mass of the pendulum bob is increased, does the clock run fast or slow?

6. Suppose we have a pendulum clock that uses only a slightly damped pendulum to keep time. The clock "ticks" each time the pendulum arm crosses $\theta = 0$.

(a) As the clock "winds down" (so the amplitude of the swings decreases), does the clock run slower or faster?

(b) If the initial push of the pendulum is large so that the pendulum swings very close to the vertical, will the clock run too fast or too slow?

7. For fixed values of b and l, for what values of the mass m will the pendulum be usable as a clock?

8. Suppose we take $l = 9.8$ m (so $g/l = 1$), $m = 1$, and b large, say $b = 4$. For the damped pendulum system above and with this choice of parameter values, do the following.

(a) Find the eigenvalues and eigenvectors of the linearized system at the equilibrium point $(0, 0)$.

(b) Find the eigenvalues and eigenvectors of the linearized system at the equilibrium point $(\pi, 0)$.

(c) Sketch the phase portrait near the equilibrium points.

(d) Sketch the entire phase portrait. [*Hint*: Begin by sketching the level sets of H as in the text.]

9. Suppose we have a pendulum clock that uses a slightly damped oscillator to keep time. Suppose that the clock "ticks" each time the pendulum arm crosses $\theta = 0$, but the arm must reach a height of $\theta = \pm 0.1$ to record the swing (that is, if the entire swing takes place with $-0.1 < \theta < 0.1$, then the clock doesn't tick). Suppose one tick is one second. In terms of the parameters b, m, and l, give a rough estimate of how long the clock can keep accurate time. Comment on why pendulum clocks must be wound.

10. **(a)** For the slightly damped pendulum ($b > 0$ but b close to zero), find the set of all initial conditions $(\theta(0), v(0))$ that execute exactly two complete revolutions for $t > 0$ (that is, pass the vertical position exactly twice) before settling into a back-and-forth swinging motion. Sketch the phase portrait for the slightly damped pendulum and shade these initial conditions.

(b) Repeat part (a) for solutions that execute exactly five complete revolutions for $t > 0$ before settling into back-and-forth swinging motion.

11. Suppose that rather than adding damping to the ideal pendulum, we add a small amount of "antidamping"; that is, we take b slightly negative in the damped pendulum system. Physically this would mean that whenever the velocity is nonzero, the pendulum is accelerated in the direction of motion.

(a) Linearize and classify the equilibrium points in this situation.

(b) Sketch the phase portrait for this system.

(c) Describe in a brief paragraph the behavior of a solution with initial condition near $\theta = v = 0$.

12. Let $G(x, y) = x^3 - 3xy^2$.

(a) What is the gradient system with vector field given by the gradient of G?

(b) Sketch the graph of G and the level sets of G.

(c) Sketch the phase portrait of the gradient system in part (a).

13. Let $G(x, y) = x^2 - y^2$.

(a) What is the gradient system with vector field given by the gradient of G?

(b) Classify the equilibrium point at the origin. [*Hint*: This system is linear.]

(c) Sketch the graph of G and the level sets of G.

(d) Sketch the phase portrait of the gradient system in part (a).

Remark: This is why saddle equilibrium points are called *saddles*.

14. Let $G(x, y) = x^2 + y^2$.

(a) What is the gradient system with vector field given by the gradient of G?

(b) Classify the equilibrium point at the origin. [*Hint*: The system is linear.]

(c) Sketch the graph of G and the level sets of G.

(d) Sketch the phase portrait of the gradient system in part (a).

15. For the "two dead fish" example given by the system

$$\frac{dx}{dt} = x - x^3$$
$$\frac{dy}{dt} = -y,$$

(a) find the linearized system for the equilibrium point at the origin and verify that the origin is a saddle,

(b) find the linearized system for the equilibrium point at $(1, 0)$ and verify that this point is a sink,

(c) from the eigenvalues and eigenvectors of the system in part (b), determine from which direction the model lobster will approach the equilibrium point $(1, 0)$, and

(d) check that the linearized system at the equilibrium point $(-1, 0)$ is the same as that at $(1, 0)$.

16. The system for the "two dead fish" example

$$\frac{dx}{dt} = x - x^3$$
$$\frac{dy}{dt} = -y$$

has the special property that the equations "decouple"; that is, the equation for dx/dt depends only on x and the equation for dy/dt depends only on y.

(a) Sketch the phase lines for the dx/dt and dy/dt equations.

(b) Using these phase lines, sketch the phase portrait of the system.

17. Suppose the smell of a bunch of dead fish in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ is given by the function

$$S(x, y) = x^2 + y^2 - \frac{x^4 + y^4}{4} - 3x^2y^2 + 100.$$

(a) What is the gradient system whose vector field is the gradient of S?

(b) Using `HPGSystemSolver`, sketch the phase portrait for this system.

(c) How many dead fish are there, and where are they?

(d) Using the results from part (b), sketch the level sets of S.

(e) Why is the model not realistic for large values of x or y?

18. A reasonable model for the smell at (x, y) of a dead fish located at (x_1, y_1) is given by

$$S_1(x, y) = \frac{1}{(x - x_1)^2 + (y - y_1)^2 + 1}.$$

That is, S_1 is given by 1 over the distance to the dead fish squared plus 1.

(a) Form the function S giving the total smell from three dead fish located at $(1, 0)$, $(-1, 0)$, and $(0, 2)$.

(b) Sketch the level sets of S.

(c) Sketch the phase portrait of the gradient system

$$\frac{dx}{dt} = \frac{\partial S}{\partial x}$$
$$\frac{dy}{dt} = \frac{\partial S}{\partial y}.$$

(d) Write out explicitly the formulas for the right-hand sides of the equations in part (c).

(e) Why did we use distance squared plus 1 instead of just distance squared in the definition of S_1? Why did we use distance squared plus 1 instead of just distance plus 1 in the definition of S_1?

19. Suppose

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

is a gradient system. That is, there exists a function $G(x, y)$ such that $f = \partial G/\partial x$ and $g = \partial G/\partial y$.

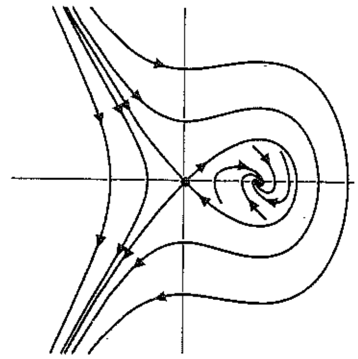
(a) Verify that if f and g have continuous partial derivatives, then $\partial f/\partial y = \partial g/\partial x$ for all (x, y).

(b) Use part (a) to show that the system

$$\frac{dx}{dt} = x^2 + 3xy$$
$$\frac{dy}{dt} = 2x + y^3$$

is *not* a gradient system.

20. The following phase portrait cannot occur for a gradient system. Why not?



21. Let

$$H(y, v) = \frac{1}{2}v^2 + V(y)$$

for some function V and consider the associated Hamiltonian system

$$\frac{dy}{dt} = \frac{\partial H}{\partial v} = v$$
$$\frac{dv}{dt} = -\frac{\partial H}{\partial y} = -\frac{dV}{dy}.$$

Let k be a positive constant. Give a physical interpretation of the relationship between the Hamiltonian system and the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\frac{dV}{dy} - kv.$$

Show that H is a Lyapunov function for this system.

22. Consider the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

and the gradient system

$$\frac{dx}{dt} = \frac{\partial H}{\partial x}$$
$$\frac{dy}{dt} = \frac{\partial H}{\partial y},$$

where H is the same function for both systems. How are the two phase portraits related?

23. Check that

$$H(x_1, x_2, p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{k_1}{2}(x_1 - L_1)^2 + \frac{k_2}{2}(x_2 - x_1 - L_2)^2.$$

is constant along solutions of

$$\frac{dx_1}{dt} = \frac{p_1}{m_1}$$

$$\frac{dx_2}{dt} = \frac{p_2}{m_2}$$

$$\frac{dp_1}{dt} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2)$$

$$\frac{dp_2}{dt} = -k_2(x_2 - x_1 - L_2).$$

24. Suppose a building with a tuned-mass damper is in an earthquake. Adapting our model from the section, we have

$$\frac{dx_1}{dt} = \frac{p_1}{m_1}$$

$$\frac{dx_2}{dt} = \frac{p_2}{m_2}$$

$$\frac{dp_1}{dt} = -k_1(x_1 - L_1) + k_2(x_2 - x_1 - L_2) + b\left(\frac{p_2}{m_2} - \frac{p_1}{m_1}\right) + A\cos\omega t$$

$$\frac{dp_2}{dt} = -k_2(x_2 - x_1 - L_2) - b\left(\frac{p_2}{m_2} - \frac{p_1}{m_1}\right),$$

where $A\cos\omega t$ is the added forcing term. How does the energy H behave with respect to time for a solution of this system for various values of k_2? First, compute the energy as a function of time, then graph this function. Use the parameter values of the example in the section ($m_1 = 1$, $k_1 = 1$, $m_2 = 0.05$, $b = 0.1$) and assume $A = 1$.

5.5 NONLINEAR SYSTEMS IN THREE DIMENSIONS

We saw in Section 2.8 that solutions of differential equations with three dependent variables are curves in three-dimensional space. These curves can loop around each other in very complicated ways. In Section 3.8 we studied the behavior of linear systems with three dependent variables. The behavior of linear systems can be determined by the