parabolic if P includes some Borel subalgebra. (In that case P is self-normalizing, by Lemma 15.2B.) Fix a base $\Delta \subset \Phi$, and set $B = B(\Delta)$. For each subset $\Delta' \subset \Delta$, define $P(\Delta')$ to be the subalgebra of L generated by all L_α ($\alpha \in \Delta$ or $-\alpha \in \Delta'$), along with H.
(a) $P(\Delta')$ is a parabolic subalgebra of L (called **standard** relative to Δ).
(b) Each parabolic subalgebra of L including $B(\Delta)$ has the form $P(\Delta')$ for some $\Delta' \subset \Delta$. [Use the Corollary of Lemma 10.2A and Proposition 8.4(d).]
(c) Prove that every parabolic subalgebra of L is conjugate under $\mathscr{E}(L)$ to one of the $P(\Delta')$.

7. Let $L = \mathfrak{sl}(2, \mathsf{F})$, with standard basis (x, h, y). For $c \in \mathsf{F}$, write $x(c) = \exp \operatorname{ad}(cx)$, $y(c) = \exp \operatorname{ad}(cy)$. Define inner automorphisms $w(c) = x(c)y(-c^{-1})x(c)$, $h(c) = w(c)w(1)^{-1}$ $(= w(c)w(-1))$, for $c \neq 0$. Compute the matrices of $w(c)$, $h(c)$ relative to the given basis of L, and deduce that all diagonal automorphisms (16.5) of L are inner. Conclude in this case that Aut L = Int $L = \mathscr{E}(L)$.
8. Let L be semisimple. Prove that the intersection of two Borel subalgebras B, B' of L always includes a CSA of L. [The proof is not easy; here is one possible outline:
(a) Let N, N' be the respective ideals of nilpotent elements in B, B'. Relative to the Killing form of L, $N = B^\perp, N' = B'^\perp$, where $\perp$ denotes orthogonal complement.
(b) Therefore $B = N^\perp = (N + (N \cap N'))^\perp = (N + (B \cap N'))^\perp = N^\perp \cap (B^\perp + N'^\perp) = B \cap (N + B') = N + (B \cap B')$.
(c) Note that $A = B \cap B'$ contains the semisimple and nilpotent parts of its elements.
(d) Let T be a maximal toral subalgebra of A, and find a T-stable complement A' to $A \cap N$. Then A' consists of semisimple elements. Since B/N is abelian, $[TA'] = 0$, forcing $A' = T$.
(e) Combine (b), (d) to obtain $B = N + T$; thus T is a maximal toral subalgebra of L.]

Notes

The proof of Theorem 16.4 is due to Winter [1] (inspired in part by G. D. Mostow); see also Barnes [1]. Most of the older proofs use analytic methods ($\mathsf{F} = \mathbf{C}$) or else some algebraic geometry: see Bourbaki [3], Chap. VII, Chevalley [2], Jacobson [1], Séminaire "Sophus Lie" [1], Serre [2]. For detailed accounts of the automorphism groups, consult Jacobson [1], Seligman [1].

Chapter V

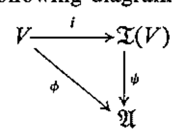
Existence Theorem

17. Universal enveloping algebras

In this section F *may be an arbitrary field (except where otherwise noted).* We shall associate to each Lie algebra L over F an associative algebra with 1 (infinite dimensional, in general), which is generated as "freely" as possible by L subject to the commutation relations in L. This "universal enveloping algebra" is a basic tool in representation theory. Although it could have been introduced right away in Chapter I, we deferred it until now in order to avoid the unpleasant task of proving the Poincaré-Birkhoff-Witt Theorem before it was really needed. The reader is advised to forget temporarily all the specialized theory of semisimple Lie algebras.

17.1. Tensor and symmetric algebras

First we introduce a couple of algebras defined by universal properties. (For further details consult, e.g., S. Lang, Algebra, Reading, Mass.: Addison-Wesley 1965, Ch. XVI.) Fix a finite dimensional vector space V over F. Let $T^0V = \mathsf{F}$, $T^1V = V$, $T^2V = V \otimes V, \ldots, T^mV = V \otimes \ldots \otimes V$ (m copies). Define $\mathfrak{T}(V) = \coprod_{i=0}^{\infty} T^iV$, and introduce an associative product, defined on homogeneous generators of $\mathfrak{T}(V)$ by the obvious rule $(v_1 \otimes \ldots \otimes v_k)(w_1 \otimes \ldots \otimes w_m) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_m \in T^{k+m}V$. This makes $\mathfrak{T}(V)$ an associative graded algebra with 1, which is generated by 1 along with any basis of V. We call it the **tensor algebra** on V. $\mathfrak{T}(V)$ is the universal associative algebra on n generators ($n = \dim V$), in the following sense: given any F-linear map $\phi: V \to \mathfrak{A}$ ($\mathfrak{A}$ an associative algebra with 1 over F), there exists a unique homomorphism of F-algebras $\psi: \mathfrak{T}(V) \to \mathfrak{A}$ such that $\psi(1) = 1$ and the following diagram commutes (i = inclusion):



Next let I be the (two sided) ideal in $\mathfrak{T}(V)$ generated by all $x \otimes y - y \otimes x$ ($x, y \in V$) and call $\mathfrak{S}(V) = \mathfrak{T}(V)/I$ the **symmetric algebra** on V; $\sigma: \mathfrak{T}(V) \to \mathfrak{S}(V)$ will denote the canonical map. Notice that the generators of I lie in T^2V; this makes it obvious that $I = (I \cap T^2V) \oplus (I \cap T^3V) \oplus \ldots$. Therefore, σ is injective on $T^0V = \mathsf{F}$, $T^1V = V$ (allowing us to identify V with a subspace of $\mathfrak{S}(V)$), and $\mathfrak{S}(V)$ inherits a grading from $\mathfrak{T}(V)$: $\mathfrak{S}(V)$

$= \coprod_{i=0}^{\infty} S^iV$. The effect of factoring out I is just to make the elements of V commute; so $\mathfrak{S}(V)$ is universal (in the above sense) for linear maps of V into commutative associative F-algebras with 1. Moreover, if $(x_1, \ldots, x_n)$ is any fixed basis of V, then $\mathfrak{S}(V)$ is canonically isomorphic to the polynomial algebra over F in n variables, with basis consisting of 1 and all $x_{i(1)} \ldots x_{i(t)}$, $t \geq 1$, $1 \leq i(1) \leq \ldots \leq i(t) \leq n$.

The reader can easily verify that the preceding constructions go through even when V is infinite dimensional.

For use much later (in §23) we mention a special fact in case char $\mathrm{F} = 0$. The symmetric group $\mathscr{S}_m$ acts on T^mV by permuting subscripts of tensors $v_1 \otimes \ldots \otimes v_m$ $(v_i \in V)$. An element of T^mV fixed by $\mathscr{S}_m$ is called a **homogeneous symmetric tensor of order m.** *Example*: $x \otimes y + y \otimes x$ (order 2). Fix a basis $(x_1, \ldots, x_n)$ of V, so the products $x_{i(1)} \otimes \ldots \otimes x_{i(m)}$ $(1 \leq i(j) \leq n)$ form a basis of T^mV. For each ordered sequence $1 \leq i(1) \leq i(2) \ldots \leq i(m) \leq n$, define a symmetric tensor

$$(*) \qquad \frac{1}{m!} \sum_{\pi \in \mathscr{S}_m} x_{i(\pi(1))} \otimes \ldots \otimes x_{i(\pi(m))}$$

(which makes sense since $m! \neq 0$ in F). The images of these tensors in S^mV are nonzero and clearly form a basis there, so the tensors (*) in turn must span a complement to $I \cap T^mV$ in T^mV. On the other hand, the tensors (*) obviously span the space of all symmetric tensors of order m (call it $\tilde{S}^mV \subset T^mV$). We conclude that σ defines a vector space isomorphism of $\tilde{S}^mV$ onto S^mV, hence of the space $\tilde{\mathfrak{S}}(V)$ of all symmetric tensors onto $\mathfrak{S}(V)$.

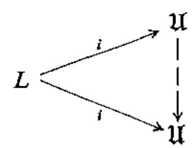
17.2. Construction of $\mathfrak{U}(L)$

We begin with the abstract definition, for an arbitrary Lie algebra L (allowed here to be infinite dimensional, contrary to our usual convention). A **universal enveloping algebra** of L is a pair $(\mathfrak{U}, i)$, where $\mathfrak{U}$ is an associative algebra with 1 over F, $i: L \rightarrow \mathfrak{U}$ is a linear map satisfying

$$(*) \qquad i([xy]) = i(x)i(y) - i(y)i(x)$$

for $x, y \in L$, and the following holds: for any associative F-algebra $\mathfrak{A}$ with 1 and any linear map $j: L \rightarrow \mathfrak{A}$ satisfying (*), there exists a unique homomorphism of algebras $\phi: \mathfrak{U} \rightarrow \mathfrak{A}$ (sending 1 to 1) such that $\phi \circ i = j$.

The *uniqueness* of such a pair $(\mathfrak{U}, i)$ is easy to prove. Given another pair $(\mathfrak{B}, i')$ satisfying the same hypotheses, we get homomorphisms $\phi: \mathfrak{U} \rightarrow \mathfrak{B}$, $\psi: \mathfrak{B} \rightarrow \mathfrak{U}$. By definition, there is a unique dotted map making the following diagram commute:



But $1_{\mathfrak{U}}$ and $\psi \circ \phi$ both do the trick, so $\psi \circ \phi = 1_{\mathfrak{U}}$. Similarly, $\phi \circ \psi = 1_{\mathfrak{B}}$.

Existence of a suitable pair $(\mathfrak{U}, i)$ is also not difficult to establish. Let $\mathfrak{T}(L)$ be the tensor algebra on L (17.1), and let J be the two sided ideal in $\mathfrak{T}(L)$ generated by all $x \otimes y - y \otimes x - [xy]$ $(x, y \in L)$. Define $\mathfrak{U}(L) = \mathfrak{T}(L)/J$, and let $\pi: \mathfrak{T}(L) \rightarrow \mathfrak{U}(L)$ be the canonical homomorphism. Notice that $J \subset \coprod_{i>0} T^iL$, so π maps $T^0L = \mathrm{F}$ isomorphically into $\mathfrak{U}(L)$ (therefore, $\mathfrak{U}(L)$ contains at least the scalars). It is not at all obvious that π maps $T^1L = L$ isomorphically into $\mathfrak{U}(L)$; this will be proved later. In any case, we claim that $(\mathfrak{U}(L), i)$ is a universal enveloping algebra of L, where $i: L \rightarrow \mathfrak{U}(L)$ is the restriction of π to L. Indeed, let $j: L \rightarrow \mathfrak{A}$ be as in the definition. The universal property of $\mathfrak{T}(L)$ yields an algebra homomorphism $\phi': \mathfrak{T}(L) \rightarrow \mathfrak{A}$ which extends j and sends 1 to 1. The special property (*) of j forces all $x \otimes y - y \otimes x - [xy]$ to lie in Ker ϕ', so ϕ' induces a homomorphism ϕ: $\mathfrak{U}(L) \rightarrow \mathfrak{A}$ such that $\phi \circ i = j$. The uniqueness of ϕ is evident, since 1 and Im i together generate $\mathfrak{U}(L)$.

Example. Let L be *abelian*. Then the ideal J above is generated by all $x \otimes y - y \otimes x$, hence coincides with the ideal I introduced in (17.1). This means that $\mathfrak{U}(L)$ coincides with the symmetric algebra $\mathfrak{S}(L)$. (In particular, $i: L \rightarrow \mathfrak{U}(L)$ is injective here.)

17.3. PBW *Theorem and consequences*

So far we know very little about the structure of $\mathfrak{U}(L)$, except that it contains the scalars. For brevity, write $\mathfrak{T} = \mathfrak{T}(L)$, $\mathfrak{S} = \mathfrak{S}(L)$, $\mathfrak{U} = \mathfrak{U}(L)$; similarly, write T^m, S^m. Define a *filtration* on $\mathfrak{T}$ by $T_m = T^0 \oplus T^1 \oplus \ldots \oplus T^m$, and let $U_m = \pi(T_m)$, $U_{-1} = 0$. Clearly, $U_mU_p \subset U_{m+p}$ and $U_m \subset U_{m+1}$. Set $G^m = U_m/U_{m-1}$ (this is just a vector space), and let the multiplication in $\mathfrak{U}$ define a bilinear map $G^m \times G^p \rightarrow G^{m+p}$. (The map is well-defined; why?) This extends at once to a bilinear map $\mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$, $\mathfrak{G} = \coprod_{m=0}^{\infty} G^m$, making $\mathfrak{G}$ a graded associative algebra with 1.

Since π maps T^m into U_m, the composite linear map $\phi_m: T^m \rightarrow U_m \rightarrow G^m = U_m/U_{m-1}$ makes sense. It is surjective, because $\pi(T_m - T_{m-1}) = U_m - U_{m-1}$. The maps ϕ_m therefore combine to yield a linear map $\phi: \mathfrak{T} \rightarrow \mathfrak{G}$, which is surjective (and sends 1 to 1).

Lemma. *$\phi: \mathfrak{T} \rightarrow \mathfrak{G}$ is an algebra homomorphism. Moreover, $\phi(I) = 0$, so ϕ induces a homomorphism ω of $\mathfrak{S} = \mathfrak{T}/I$ onto $\mathfrak{G}$.*

Proof. Let $x \in T^m$, $y \in T^p$ be homogeneous tensors. By definition of the product in $\mathfrak{G}$, $\phi(xy) = \phi(x)\phi(y)$, so it follows that ϕ is multiplicative on $\mathfrak{T}$. Let $x \otimes y - y \otimes x$ $(x, y \in L)$ be a typical generator of I. Then $\pi(x \otimes y - y \otimes x) \in U_2$, by definition. On the other hand, $\pi(x \otimes y - y \otimes x) = \pi([xy]) \in U_1$, whence $\phi(x \otimes y - y \otimes x) \in U_1/U_1 = 0$. It follows that $I \subset \mathrm{Ker}\ \phi$. ☐

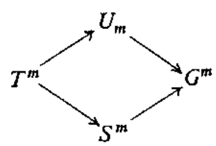
The following theorem is the basic result about $\mathfrak{U}(L)$; it (or its Corollary

C) is called the **Poincaré-Birkhoff-Witt Theorem** (or PBW Theorem). The proof will be given in (17.4).

Theorem. *The homomorphism* $\omega: \mathfrak{S} \to \mathfrak{G}$ *is an isomorphism of algebras.*

Corollary A. *Let* W *be a subspace of* T^m*. Suppose the canonical map* $T^m \to S^m$ *sends* W *isomorphically onto* S^m*. Then* $\pi(W)$ *is a complement to* U_{m-1} *in* U_m*.*

Proof. Consider the diagram (all maps canonical):



Thanks to the lemma above (and the definitions), this is a commutative diagram. Since $\omega: \mathfrak{S} \to \mathfrak{G}$ is an isomorphism (by the theorem), the bottom map sends $W \subset T^m$ isomorphically onto G^m. Reverting to the top map, we get the corollary. ▯

Corollary B. *The canonical map* $i: L \to \mathfrak{U}(L)$ *is injective (so* L *may be identified with* $i(L)$*).*

Proof. This is the special case $W = T^1$ $(=L)$ of Corollary A. ▯

We have allowed L to be infinite dimensional. In practice, the case where L has countable basis is quite adequate for our purposes.

Corollary C. *Let* $(x_1, x_2, x_3, \ldots)$ *be any ordered basis of* L*. Then the elements* $x_{i(1)} \ldots x_{i(m)} = \pi(x_{i(1)} \otimes \ldots \otimes x_{i(m)})$, $m \in \mathbf{Z}^+$, $i(1) \leq i(2) \ldots \leq i(m)$*, along with* 1*, form a basis of* $\mathfrak{U}(L)$*.*

Proof. Let W be the subspace of T^m spanned by all $x_{i(1)} \otimes \ldots \otimes x_{i(m)}$, $i(1) \leq \ldots \leq i(m)$. Evidently W maps isomorphically onto S^m, so Corollary A shows that $\pi(W)$ is a complement to U_{m-1} in U_m. ▯

A basis of $\mathfrak{U}(L)$ of the type just constructed will be referred to simply as a **PBW basis.**

Corollary D. *Let* H *be a subalgebra of* L*, and extend an ordered basis* $(h_1, h_2, \ldots)$ *of* H *to an ordered basis* $(h_1, \ldots, x_1, \ldots)$ *of* L*. Then the homomorphism* $\mathfrak{U}(H) \to \mathfrak{U}(L)$ *induced by the injection* $H \to L \to \mathfrak{U}(L)$ *is itself injective, and* $\mathfrak{U}(L)$ *is a free* $\mathfrak{U}(H)$*-module with free basis consisting of all* $x_{i(1)} \ldots x_{i(m)}$, $i(1) \leq i(2) \leq \ldots \leq i(m)$*, along with* 1.

Proof. These assertions follow at once from Corollary C. ▯

For use much later, we record a special fact.

Corollary E. *Let* char $\mathsf{F} = 0$*. With notation as in (17.1), the composite* $S^mL \to \tilde{S}^mL \to U_m$ *of canonical maps is a (linear) isomorphism of* S^mL *onto a complement of* U_{m-1} *in* U_m*.*

Proof. Use Corollary A, with $W = \tilde{S}^m$. ▯

17.4. Proof of PBW *Theorem*

Fix an ordered basis $(x_\lambda; \lambda \in \Omega)$ of L. This choice identifies $\mathfrak{S}$ with the polynomial algebra in indeterminates z_λ $(\lambda \in \Omega)$. For each sequence $\Sigma = (\lambda_1, \ldots, \lambda_m)$ of indices (m is called the *length* of Σ), let $z_\Sigma = z_{\lambda_1} \ldots z_{\lambda_m} \in S^m$ and let $x_\Sigma = x_{\lambda_1} \otimes \ldots \otimes x_{\lambda_m} \in T^m$. Call Σ *increasing* if $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$, in the given ordering of Ω; by fiat, $\varnothing$ is increasing and $z_\varnothing = 1$. So $\{z_\Sigma | \Sigma$ increasing$\}$ is a basis of $\mathfrak{S}$. Associated with the grading $\mathfrak{S} = \amalg S^m$ is the filtration $S_m = S^0 \oplus \ldots \oplus S^m$. In the following lemmas, write $\lambda \leq \Sigma$ if $\lambda \leq \mu$ for all $\mu \in \Sigma$.

Lemma A. *For each* $m \in \mathbf{Z}^+$*, there exists a unique linear map* $f_m: L \otimes S_m \to \mathfrak{S}$ *satisfying:*

(A_m) $f_m(x_\lambda \otimes z_\Sigma) = z_\lambda z_\Sigma$ for $\lambda \leq \Sigma$, $z_\Sigma \in S_m$.

(B_m) $f_m(x_\lambda \otimes z_\Sigma) - z_\lambda z_\Sigma \in S_k$ for $k \leq m$, $z_\Sigma \in S_k$.

(C_m) $f_m(x_\lambda \otimes f_m(x_\mu \otimes z_T)) = f_m(x_\mu \otimes f_m(x_\lambda \otimes z_T)) + f_m([x_\lambda x_\mu] \otimes z_T)$ *for all* $z_T \in S_{m-1}$.

Moreover, the restriction of f_m *to* $L \otimes S_{m-1}$ *agrees with* f_{m-1}*.*

Proof. Notice that the terms in (C_m) all make sense, once (B_m) is proved. Notice too that the restriction of f_m to $L \otimes S_{m-1}$ automatically satisfies (A_{m-1}), (B_{m-1}), (C_{m-1}), so this restricted map must coincide with f_{m-1} because of the asserted uniqueness. To verify existence and uniqueness of f_m, we proceed by induction on m. For $m = 0$, only $z_\Sigma = 1$ occurs; therefore we may let $f_0(x_\lambda \otimes 1) = z_\lambda$ (and extend linearly to $L \otimes S_0$). Evidently (A_0), (B_0), (C_0) are satisfied, and moreover, (A_0) shows that our choice of f_0 is the only possible one.

Assuming the existence of a unique f_{m-1} satisfying (A_{m-1}), (B_{m-1}), (C_{m-1}), we shall show how to extend f_{m-1} to a map f_m. For this it will suffice to define $f_m(x_\lambda \otimes z_\Sigma)$ when Σ is an increasing sequence of length m.

For the case $\lambda \leq \Sigma$, (A_m) cannot hold unless we define $f_m(x_\lambda \otimes z_\Sigma) = z_\lambda z_\Sigma$. In case $\lambda \leq \Sigma$ fails, the first index μ in Σ must be strictly less than λ, so $\Sigma = (\mu, T)$, where of course $\mu \leq T$ and T has length $m-1$. By (A_{m-1}), $z_\Sigma = z_\mu z_T = f_{m-1}(x_\mu \otimes z_T)$. Since $\mu \leq T$, $f_m(x_\mu \otimes z_T) = z_\mu z_T$ is already defined, so the left side of (C_m) becomes $f_m(x_\lambda \otimes z_\Sigma)$. On the other hand, (B_{m-1}) implies that $f_m(x_\lambda \otimes z_T) = f_{m-1}(x_\lambda \otimes z_T) \equiv z_\lambda z_T \pmod{S_{m-1}}$. This shows that the right side of (C_m) is already defined:

$$z_\mu z_\lambda z_T + f_{m-1}(x_\mu \otimes y) + f_{m-1}([x_\lambda x_\mu] \otimes z_T), \; y \in S_{m-1}.$$

The preceding remarks show that f_m can be defined, and in only one way. Moreover, (A_m) and (B_m) clearly hold, as does (C_m) in case $\mu < \lambda$, $\mu \leq T$. But $[x_\mu x_\lambda] = -[x_\lambda x_\mu]$, so (C_m) also holds for $\lambda < \mu$, $\lambda \leq T$. When $\lambda = \mu$, (C_m) is also true. It remains only to consider the case where neither $\lambda \leq T$ nor $\mu \leq T$ is true. Write $T = (\nu, \Psi)$, where $\nu \leq \Psi$, $\nu < \lambda$, $\nu < \mu$. To keep the notation under control, abbreviate $f_m(x \otimes z)$ by xz whenever $x \in L$, $z \in S_m$.

The induction hypothesis insures that $x_\mu z_T = x_\mu(x_\nu z_\Psi) = x_\nu(x_\mu z_\Psi)+[x_\mu x_\nu]z_\Psi$, and $x_\mu z_\Psi = z_\mu z_\Psi + w$ ($w \in S_{m-2}$), by (B_{m-2}). Since $\nu \leq \Psi$, $\nu < \mu$, (C_m) applies already to $x_\lambda(x_\nu(z_\mu z_\Psi))$. By induction, (C_m) also applies to $x_\lambda(x_\nu w)$, therefore to $x_\lambda(x_\nu(x_\mu z_\Psi))$. Consequently: (*) $x_\lambda(x_\mu z_T) = x_\nu(x_\lambda(x_\mu z_\Psi))+[x_\lambda x_\nu]\,(x_\mu z_\Psi)+[x_\mu\, x_\nu]\,(x_\lambda z_\Psi)+[x_\lambda\,[x_\mu\, x_\nu]]z_\Psi$.

Recall that λ, μ are interchangeable throughout this argument. If we interchange them in (*) and subtract the two resulting equations, we get:

$$\begin{aligned}x_\lambda(x_\mu z_T)-x_\mu(x_\lambda z_T) &= x_\nu(x_\lambda(x_\mu z_\Psi))-x_\nu(x_\mu(x_\lambda z_\Psi))+[x_\lambda\,[x_\mu x_\nu]]z_\Psi -\\ &-[x_\mu\,[x_\lambda x_\nu]]z_\Psi = x_\nu([x_\lambda x_\mu]z_\Psi)+[x_\lambda\,[x_\mu x_\nu]]z_\Psi\\ &+[x_\mu\,[x_\nu x_\lambda]]z_\Psi = [x_\lambda x_\mu]\,(x_\nu z_\Psi)+([x_\nu\,[x_\lambda x_\mu]]\\ &+[x_\lambda\,[x_\mu x_\nu]]+[x_\mu\,[x_\nu x_\lambda]])z_\Psi = [x_\lambda x_\mu]z_T\end{aligned}$$

(thanks to the Jacobi identity).

This proves (C_m), and with it the lemma. ▯

Lemma B. *There exists a representation* ρ: $L \rightarrow \mathfrak{gl}(\mathfrak{S})$ *satisfying:*

(*a*) $\rho(x_\lambda)z_\Sigma = z_\lambda z_\Sigma$ *for* $\lambda \leq \Sigma$.

(*b*) $\rho(x_\lambda)z_\Sigma \equiv z_\lambda z_\Sigma \pmod{S_m}$, *if* Σ *has length m.*

Proof. Lemma A allows us to define a linear map f: $L \otimes \mathfrak{S} \rightarrow \mathfrak{S}$ satisfying (A_m), (B_m), (C_m) for all m (since f_m restricted to $L \otimes S_{m-1}$ is f_{m-1}, by the uniqueness part). In other words $\mathfrak{S}$ becomes an L-module (condition (C_m)), affording a representation ρ which satisfies (a), (b), thanks to (A_m), (B_m). ▯

Lemma C. *Let* $t \in T_m \cap J$ (J = *Ker* π, π: $\mathfrak{T} \rightarrow \mathfrak{U}$ *canonical*). *Then the homogeneous component* t_m *of t of degree m lies in I* (*the kernel of the canonical map* $\mathfrak{T} \rightarrow \mathfrak{S}$).

Proof. Write t_m as linear combination of basis elements $x_{\Sigma(i)}$ ($1 \leq i \leq r$), each $\Sigma(i)$ of length m. The Lie homomorphism ρ: $L \rightarrow \mathfrak{gl}(\mathfrak{S})$ constructed in Lemma B extends, by the universal property of $\mathfrak{U}$, to an algebra homomorphism (also called ρ) $\mathfrak{T} \rightarrow \text{End}\ \mathfrak{S}$, with $J \subset \text{Ker}\ \rho$. So $\rho(t) = 0$. But $\rho(t).1$ is a polynomial whose term of highest degree is the appropriate combination of the $z_{\Sigma(i)}$ ($1 \leq i \leq r$), by Lemma B. Therefore this combination of the $z_{\Sigma(i)}$ is 0 in $\mathfrak{S}$, and $t_m \in I$ as required. ▯

Proof of PBW *Theorem.* Let $t \in T^m$, π: $\mathfrak{T} \rightarrow \mathfrak{U}$ the canonical map. We must show that $\pi(t) \in U_{m-1}$ implies $t \in I$. But $t \in T^m$, $\pi(t) \in U_{m-1}$ together imply that $\pi(t) = \pi(t')$ for some $t' \in T_{m-1}$, whence $t - t' \in J$. Apply Lemma C to the tensor $t-t' \in T_m \cap J$: the homogeneous component of degree m being t, we get $t \in I$. ▯

17.5. Free Lie algebras

The reader may be familiar with the method of constructing groups by generators and relations. We shall use an analogous method in §18 to construct semisimple Lie algebras. For this one needs the notion of free Lie algebra.

Let L be a Lie algebra over F generated by a set X. We say L is **free on X** if, given any mapping ϕ of X into a Lie algebra M, there exists a unique homomorphism ψ: $L \rightarrow M$ extending ϕ. The reader can easily verify the *uniqueness* (up to a unique isomorphism) of such an algebra L. As to its *existence*, we begin with a vector space V having X as basis, form the tensor algebra $\mathfrak{T}(V)$ (viewed as Lie algebra via the bracket operation), and let L be the Lie subalgebra of $\mathfrak{T}(V)$ generated by X. Given any map ϕ: $X \rightarrow M$, let ϕ be extended first to a linear map $V \rightarrow M \subset \mathfrak{U}(M)$, then (canonically) to an associative algebra homomorphism $\mathfrak{T}(V) \rightarrow \mathfrak{U}(M)$, or a Lie homomorphism (whose restriction to L is the desired ψ: $L \rightarrow M$, since ψ maps the generators X into M).

We remark that if L is free on a set X, then a vector space V can be given an L-module structure simply by assigning to each $x \in X$ an element of the Lie algebra $\mathfrak{gl}(V)$ and extending canonically.

Finally, if L is free on X, and if R is the ideal of L generated by elements f_j (j running over some index set), we call L/R the Lie algebra with **generators** x_i and **relations** $f_j = 0$, where x_i are the images in L/R of the elements of X.

Exercises

1. Prove that if dim $L < \infty$, then $\mathfrak{U}(L)$ has no zero divisors. [Hint: Use the fact that the associated graded algebra $\mathfrak{G}$ is isomorphic to a polynomial algebra.]
2. Let L be the two dimensional nonabelian Lie algebra (1.4), with $[xy] = x$. Prove directly that i: $L \rightarrow \mathfrak{U}(L)$ is injective (i.e., that $J \cap L = 0$).
3. If $x \in L$, extend ad x to an endomorphism of $\mathfrak{U}(L)$ by defining ad $x(y) = xy-yx$ ($y \in \mathfrak{U}(L)$). If dim $L < \infty$, prove that each element of $\mathfrak{U}(L)$ lies in a finite dimensional L-submodule. [If $x, x_1, \ldots, x_m \in L$, verify that ad $x(x_1 \ldots x_m) = \sum_{i=1}^{m} x_1 x_2 \ldots \text{ad}\ x(x_i) \cdots x_m$.]
4. If L is a free Lie algebra on a set X, prove that $\mathfrak{U}(L)$ is isomorphic to the tensor algebra on a vector space having X as basis.
5. Describe the free Lie algebra on a set $X = \{x\}$.
6. How is the PBW Theorem used in the construction of free Lie algebras?

Notes

Our treatment of the PBW Theorem follows Bourbaki [1]. For another approach, see Jacobson [1].

18. Generators and relations

We can now resume our study of a semisimple Lie algebra L over the algebraically closed field F of characteristic 0. The object is to find a pre-