Lecture 10 - Representation Theory III: Theory of Weights

February 18, 2012

1 Terminology

One assumes a base $\triangle = \{\alpha_i\}_i$ has been chosen. Then a weight Λ with non-negative integral Dynkin coefficients $\Lambda_i = \langle \Lambda, \alpha_i \rangle$ is called a *dominant weight*. If all coefficients are all positive, it is called *strongly dominant*.

The element

$$\delta \;=\; \frac{1}{2} \sum_{\alpha > 0} \alpha \tag{1}$$

recurs frequently enough that we give it a name, the *Weyl vector*. Its weight is $(1, \ldots, 1)$ and lies in the fundamental Weyl chamber.

If $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ is a weight, it components

$$\Lambda_i \;=\; \langle \Lambda,\, \alpha_i \rangle \;=\; \frac{2\,(\Lambda,\, \alpha_i)}{(\alpha_i,\, \alpha_i)} \tag{2}$$

are called its Dynkin coefficients. The vectors

$$\lambda^i \;=\; (0, \ldots, 1, \ldots, 0) \tag{3}$$

(a 1 in the i^{th} position) are then dual to the vectors $2\alpha_i/(\alpha_i, \alpha_i)$, in the sense that

$$\left(\Lambda^i,\, \frac{2\alpha_j}{(\alpha_j,\, \alpha_j)} \right) \;=\; \delta^i_j \tag{4}$$

Weights are $\mathbb{Z}$-linear combinations of the λ^i, so we can produce nice visualizations of representations, at least in low-dimensional cases.

2 Automorphisms

Lie algebra automorphisms produce automorphisms of their representations. Recall that the automorphism group of a semisimple algebra $\mathfrak{g}$ has the form $Out \ltimes Inn$, inner times outer automorphisms. The inner automorphisms consist of Weyl transformations (or re-selection of the base) and diagonal transformations, and the outer automorphisms are permutations of a fixed base (and correspond to the Dynkin diagram automorphisms).

The diagonal automorphisms have no effect on the weight schemes. That is they don't affect what vectors go to which, they just alter coefficients.

Any Weyl transformation re-orders the weights in a given representation. Any weights conjugate to a highest weight under some Weyl transformation is called an *extreme* weight.

Outer automorphism permute representations.

3 Partial Order and Height

Given two weights λ, μ, we say that $\lambda > \mu$ if $\lambda - \mu$ is not zero and is a non-negative sum of positive weights. That is, $\lambda - \mu$ is nonzero, but in the positive cone of $\triangle$. This is not the same as saying $\lambda - \mu$ is dominant, which would mean it is a non-negative positive sum of positive fundamental weights. If $\lambda - \mu$ is neither in the positive cone nor in the negative cone of $\triangle$, the weights are *incomeasurable*.

Given a representation, it is possible to define the *height* (or *length*) of a weight vector. If $v_0 \in V_\Lambda$ is a highest weight vector and $y_{i_1} \dots y_{i_m}.v_0 = 0$ (not necessarily ordered) where m is minimal, we say v_0 has height $\frac{1}{2}(m-1)$. The height of any other weight vector v is $m-n$ where $v = y_{j_1} \dots y_{j_n}.v_0$ and n is minimal (again, the monomial is not necessarily ordered). Given a representation, heights are not necessarily integral, but are uniquely defined, which is due to the fact that $\triangle$ is a basis of $\mathfrak{h}^*$.

In the case of the adjoint representation, this corresponds to the previously defined height of a root.

4 Casimir Operator

Let $\triangle = \{\alpha_i\}_i$ be the simple roots of some semi-simple algebra with corresponding $\{h_i, x_i, y_i\}$. Define coefficients $\kappa_{ij} = \kappa(h_i, h_j)$, and define the matrix (κ^{ij}) to be the inverse of (κ_{ij}). Recalling that $\kappa(x_i, y_j) = \left(\frac{2}{(\alpha_i, \alpha_i)}\right)^{-1} \delta_{ij}$ and $\kappa(x_i, x_j) = \kappa(y_i, y_j) = 0$, we can write the

Casimir element

$$c \;=\; \sum_{i,j} \kappa^{ij} h_i h_j \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} (x_\alpha y_\alpha \;+\; y_\alpha x_\alpha)\,. \tag{5}$$

We know that $c = const$ on V^Λ. To compute the value of this constant, let $v_0 \in V_\Lambda$ be the highest weight. Since $x_\alpha . v_0 = 0$ we have

$$\begin{aligned}
c.v_0 \;&=\; \sum_{i,j} \kappa^{ij} h_i h_j . v_0 \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} (x_\alpha y_\alpha \;+\; y_\alpha x_\alpha) \\
&=\; \sum_{i,j} \kappa^{ij} \,(\Lambda, h_i)\,(\Lambda, h_j)\, v_0 \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} x_\alpha y_\alpha . v_0 \\
&=\; (\Lambda,\, \Lambda)\, v_0 \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} ([x_\alpha, y_\alpha] \;+\; y_\alpha x_\alpha)\,. v_0 \\
&=\; (\Lambda,\, \Lambda)\, v_0 \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} 2 h_\alpha . v_0 \\
&=\; (\Lambda,\, \Lambda)\, v_0 \;+\; \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)} \right)^{-1} \frac{2}{(\alpha,\alpha)}\,(\Lambda,\, \alpha)\; v_0 \\
&=\; \left((\Lambda,\, \Lambda) \;+\; \sum_{\alpha>0} (\Lambda,\, \alpha) \right)\; v_0
\end{aligned} \tag{6}$$

so that

$$c\big|_{V^\Lambda} \;=\; (\Lambda,\, \Lambda) \;+\; 2\,(\Lambda,\, \delta) \tag{7}$$

5 The Casimir on $\mathfrak{sl}(2, \mathbb{C})$

Now on $\mathfrak{sl}_2$ we have the adjoint representation, where $\{h, x, y\}$ is the basis, and $ad_h x = \alpha(h)x$, where α is the root and $\alpha(h) = 2$. With

$$\kappa = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \tag{8}$$

we have $(h, h) = 8$, so that with $h_\flat = \frac{2\alpha}{(\alpha,\alpha)}$ we have $8 = \frac{4}{(\alpha,\alpha)}$ so $(\alpha, \alpha) = \frac{1}{2}$. The Casimir is then

$$c \;=\; \frac{1}{8} h.h. \;+\; \frac{1}{4}\,(x.y. \;+\; y.x.) \tag{9}$$

Let V^Λ be the irreducible weight space of highest weight Λ, which is just a non-negative integer. Then Λ is characterized by a single Dynkin coefficient $\Lambda_1 = \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, so $\Lambda = \frac{1}{2}\Lambda_1\alpha$. Then

$$(\Lambda,\, \Lambda) \;=\; \frac{1}{4}\Lambda_1^2\,(\alpha,\alpha) \;=\; \frac{1}{8}\Lambda_1^2. \tag{10}$$

The Weyl symbol δ is $\frac{1}{2}\alpha$, so that

$$c \;=\; (\Lambda,\, \Lambda + \alpha) \;=\; \frac{1}{8}\Lambda_1^2 + \frac{1}{4}\Lambda_1 \;=\; \frac{1}{8}\Lambda_1\,(\Lambda_1 + 2)\,. \tag{11}$$

6 Freudenthal's Dimension Formula

Consider the weight space of weight μ in V_μ^Λ, which we denote V_μ^Λ. If the dimension of the weight space is N_μ, and we trace the Casimir over this weight space we obtain

$$Tr\; c\big|_{V_\mu^\Lambda} \;=\; N_\mu\;(\Lambda,\, \Lambda + 2\delta) \tag{12}$$

On the other hand we can attempt a direct calculation of the action of c on V_μ^Λ. With

$$c \;=\; \sum_{i,j}\kappa^{ij}h_ih_j \;+\; \sum_{\alpha>0}\left(\frac{2}{(\alpha,\,\alpha)}\right)^{-1}(x_\alpha\, y_\alpha \;+\; y_\alpha\, x_\alpha) \tag{13}$$

The first part is easy:

$$\sum_{i,j}\kappa^{ij}h_ih_j.v_\mu \;=\; \sum_{i,j}\kappa^{ij}h_i\,(\mu,\,h_j)\,.v_\mu \;=\; \sum_{i,j}\kappa^{ij}\,(\mu,\,h_i)\,(\mu,\,h_j)\,.v_\mu \tag{14}$$

$$=\; (\mu,\,\mu)\;v_\mu \tag{15}$$

Thus

$$Tr\left(\sum_{i,j}\kappa^{ij}h_ih_j\right)\;\Big|_{V_\mu^\Lambda} \;=\; (\mu,\,\mu)\,N_\mu \tag{16}$$

We have to compute the value of the second term.

Now any vector $v_\mu \in V_\mu^\Lambda$ is in the middle of the α-weight string of highest weight t. Restricting to this $\mathfrak{sl}_2$ representation, the Casimir is

$$\begin{aligned}
c \;&=\; \frac{1}{8}h_\alpha h_\alpha \;+\; \frac{1}{4}\,(x_\alpha y_\alpha \;+\; y_\alpha x_\alpha)\\
\frac{1}{8}t\,(t+2)\;v_m \;&=\; \frac{1}{8}h_\alpha h_\alpha.v_m \;+\; \frac{1}{4}\,(x_\alpha y_\alpha \;+\; y_\alpha x_\alpha)\,.v_m\\
&=\; \frac{1}{8}\frac{2}{(\alpha,\alpha)}\,(m,\alpha)\,h_\alpha.v_m \;+\; \frac{1}{4}\,(x_\alpha y_\alpha \;+\; y_\alpha x_\alpha)\,.v_m\\
&=\; \frac{1}{8}\left(\frac{2}{(\alpha,\alpha)}\right)^2(m,\alpha)^2\;v_m \;+\; \frac{1}{4}\,(x_\alpha y_\alpha \;+\; y_\alpha x_\alpha)\,.v_m
\end{aligned} \tag{17}$$

Then

$$(x_\alpha y_\alpha \,+\, y_\alpha x_\alpha)\,.v_\mu \;=\; \frac{1}{2}t\,(t+2)\;v_\mu \,-\, \frac{2\,(\mu,\alpha)^2}{(\alpha,\alpha)^2}\,v_\mu. \tag{18}$$

It will be convenient to re-express this in a slightly different way. There is a non-negative integer k so that $\mu + k\alpha$ is the highest weight. We have

$$t\,v_{\mu+k\alpha} = h_\alpha.v_{\mu+k\alpha} \;=\; \left(2\frac{(\mu,\alpha)}{(\alpha,\alpha)} \,+\, 2k\right)\,v_{\mu+k\alpha} \tag{19}$$

so that

$$\begin{aligned}(x_\alpha y_\alpha \,+\, y_\alpha x_\alpha)\,.v_\mu \;&=\; \frac{2\,((\mu,\alpha) \,+\, k(\alpha,\alpha))\,((\mu,\alpha) \,+\, (k+1)(\alpha,\alpha))}{(\alpha,\alpha)^2}\,v_\mu \,-\, \frac{2\,(\mu,\alpha)^2}{(\alpha,\alpha)^2}\,v_\mu \\ &=\; \frac{2(2k+1)\,(\mu,\,\alpha) \,+\, 2k(k+1)(\alpha,\alpha)}{(\alpha,\alpha)}\,v_\mu\end{aligned} \tag{20}$$

On an individual vector we have

$$\sum_{\alpha>0}\left(\frac{2}{(\alpha,\alpha)}\right)^{-1}(x_\alpha y_\alpha \,+\, y_\alpha x_\alpha)\,.v_\mu \;=\; \sum_{\alpha>0}\left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right)\,v_\mu \tag{21}$$

so that

$$\begin{aligned}&Tr\;\sum_{\alpha>0}\left(\frac{2}{(\alpha,\alpha)}\right)^{-1}(x_\alpha y_\alpha \,+\, y_\alpha x_\alpha) \\ &\quad=\;\sum_{\alpha>0} dim^\Lambda_\mu(k)\,\left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right)\end{aligned} \tag{22}$$

where $dim(\mu,k)$ is the dimension of the subspace of V^Λ_μ spanned by those vectors within $\mathfrak{sl}_2$-representations of highest weight $\mu + k\alpha$. Thus

$$dim(\mu,k) \;=\; N_{\mu+k\alpha} \,-\, N_{\mu+(k+1)\alpha} \tag{23}$$

so that

$$
\begin{aligned}
& Tr \sum_{\alpha>0} \left(\frac{2}{(\alpha,\alpha)}\right)^{-1} (x_\alpha y_\alpha \,+\, y_\alpha x_\alpha) \\
&\quad = \sum_{\alpha>0}\sum_{k=0}^{\infty} \left(N_{\mu+k\alpha} - N_{\mu+(k+1)\alpha}\right) \left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right) \\
&\quad = \sum_{\alpha>0}\sum_{k=0}^{\infty} N_{\mu+k\alpha} \left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right) \\
&\qquad - \sum_{\alpha>0}\sum_{k=0}^{\infty} N_{\mu+(k+1)\alpha} \left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right) \\
&\quad = \sum_{\alpha>0}\sum_{k=0}^{\infty} N_{\mu+k\alpha} \left((2k+1)\,(\mu,\,\alpha) \,+\, k(k+1)(\alpha,\alpha)\right) \\
&\qquad - \sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((2k-1)\,(\mu,\,\alpha) \,+\, (k-1)k(\alpha,\alpha)\right) \\
&\quad = \sum_{\alpha>0} N_\mu\,(\mu,\,\alpha) \,+\, \sum_{\alpha>0} 2\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right) \\
&\quad = 2N_\mu\,(\mu,\,\delta) \,+\, 2\sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right)
\end{aligned}
\tag{24}
$$

Bringing it together, we have

$$
\begin{aligned}
N_\mu\,(\Lambda,\,\Lambda+2\delta) \;&=\; N_\mu\,(\mu,\,\mu) \,+\, 2N_\mu\,(\mu,\,\delta) \,+\, 2\sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right) \\
N_\mu\left((\Lambda,\,\Lambda+2\delta) - (\mu,\,\mu+2\delta)\right) \;&=\; 2\sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right) \\
N_\mu\,(\Lambda+\mu+2\delta,\,\Lambda-\mu) \;&=\; 2\sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right)
\end{aligned}
\tag{25}
$$

We arrive at Freudenthal's formula:

$$
N_\mu \;=\; \frac{2\sum_{\alpha>0}\sum_{k=1}^{\infty} N_{\mu+k\alpha} \left((\mu,\,\alpha) \,+\, k(\alpha,\alpha)\right)}{(\Lambda+\mu+2\delta,\,\Lambda-\mu)} \tag{26}
$$