Lecture 1 - Lie Groups and the Maurer-Cartan equation

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1 Lie groups

A Lie group is a differentiable manifold along with a group structure so that the group operations of products and inverses are differentiable.

Let $G$ be a Lie group. An element $A \in G$ induces three standard diffeomorphisms

$$L_A, R_A, \text{Ad}_A : G \to G$$

(1)

namely left translation, right translation, and the adjoint transformation $\text{Ad}_A = L_AR_A$.

Note that $L_AR_B = R_BL_A$, so left and right translation commute.

A Lie group has a standard element, the identity $e \in G$. Given any vector $v \in T_eG$, we can use left translation (right translation, if we wanted) to produce a global vector field by

$$v_A \in T_AG, \quad v_A = L_A^*v$$

(2)

where $L_A^*T_eG \to T_AG$ is the push-forward along the diffeomorphism $L_A$. Any vector field $v$ so that

$$v_{BA} = L_B^*v_A, \quad v_{BA} \in T_BAG, \quad v_A \in TVG$$

(3)

is called left-invariant (similarly for right-invariant vector fields). Because $L_A^*L_B^* = L_{AB}^*$, such fields are well defined, and determined by a vector at any point of the manifold.

Given $v, w \in T_eG$, after extending them to left-invariant fields, we define the Lie bracket of the two vectors to be the topological bracket of the left-invariant fields, restricted to the vector at $e$. Since $L_A$ is a diffeomorphism, we have

$$L_A^*[v, w] = [L_A^*v, L_A^*w]$$

(4)

for any vector fields, so that the bracket of two left-invariant fields is a left-invariant field. It is thus convenient to identify the algebra on the tangent space at the identity with the
algebra of left-invariant vector fields on the manifold $G$. Since this is a Lie subalgebra of the Lie algebra of all differentiable vector fields under the bracket, the Jacobi identity and antisymmetry hold, so we have a Lie algebra $\mathfrak{g}$ canonically associated with the group $G$, with $\dim \mathfrak{g} = \dim G$.

We have two canonical representations on $\mathfrak{g}$. The first is the standard adjoint representation $\text{ad}_v : \mathfrak{g} \to \mathfrak{g}$ given by $\text{ad}_v w = [v, w]$. Note that given any $A \in G$ we have $\text{Ad}_A^* : T_e G \to T_e G$, which is a linear map. To see this is a representation, first extend $v, w \in T_e G$ to left-invariant fields. Note that $\text{Ad}_A = L_A^* R_A^*$, that $L_A^*$ leaves any left-invariant field invariant, and $R_A^*$ takes left-invariant fields to left-invariant fields (by the commutativity of $L_B$ (any $B$) and $R_A^*$). Finally since $\text{Ad}_A : G \to G$ is a diffeomorphism

$$\text{Ad}_A^*[v, w] = [\text{Ad}_A^* v, \text{Ad}_A^* w] \quad (5)$$

It is customary to drop the $*$ and just write $\text{Ad}_A$, with context distinguishing whether it is a map $T_e G \to T_e G$ or $G \to G$.

2 Left-fields, Right-actions, and exponentials

2.1 Basics

Any vector $v \in \mathfrak{g} \approx T_e G$ extends uniquely to a left-invariant vector field, which integrates out to a 1-parameter family of diffeomorphisms

$$\varphi_t : G \to G \quad (6)$$

which has the property that

$$\varphi_t(A) = L_A \varphi_t(e) = R_{\varphi_t(e)^{-1}} A. \quad (7)$$

which is simply the integrated version of $v_A = L_A^* v$. Therefore left-invariant fields integrate out to right-translations, or, put more accurately, the flow of a left-invariant vector field is a flow by right translations. We also have the group law

$$R_{\varphi_{s+t}(e)}^{-1} A = \varphi_{t+s}(A) = \varphi_t(\varphi_s(A)) = R_{\varphi_t(e)^{-1}} R_{\varphi_s(e)^{-1}} A \quad (8)$$

so that $\varphi_t(e) \varphi_s(e) = \varphi_{s+t}(e)$ (note this also implies $\varphi_{-t}(e) = \varphi_t(e)^{-1}$).

Similarly, and flow by a 1-parameter subgroup of right translations determines a left-invariant vector field.

The path

$$\varphi_t(e) \quad (9)$$

through the origin is called the exponential map, denoted $exp(tv)$. It obeys the group law

$exp(tv)exp(sv) = exp((t + s)v)$, as shown above.
2.2 Adjoint actions

Consider the differential of the adjoint action along $\exp(tv)$ on the vector $w$

$$\frac{d}{dt} \bigg|_{t=0} Ad_{\exp(tv)} w = \frac{d}{dt} \bigg|_{t=0} L^*_{\exp(tv)} R^*_{\exp(tv)} w$$

(10)

A vector is determined by its action on functions, and this action is given by

$$\frac{d}{ds} \bigg|_{s=0} \frac{d}{dt} \bigg|_{t=0} f(Ad_{exp(tv)exp(sw)})$$

$$= \frac{d}{ds} \bigg|_{s=0} \frac{d}{dt} \bigg|_{t=0} \left( L_{\exp((t+\tau)v)} L_{\exp(sw)} L_{\exp(-tv)} f(e) \right)$$

$$= L_{\exp(\tau v)} R_{\exp(\tau v)} \frac{d}{ds} \bigg|_{s=0} \frac{d}{dt} \bigg|_{t=0} \left( L_{\exp(sw)} L_{\exp(-tv)} f(e) \right)$$

$$= L_{\exp(\tau v)} R_{\exp(\tau v)} \left( v(L_{\exp(sw)} f)(e) - L_{\exp(sw)} (vf)(e) \right)$$

so that

$$\frac{d}{dt} Ad_{\exp(tv)} = Ad_{\exp(tv)} \circ ad_v$$

(12)

Note in particular that

$$\frac{d}{dt} \bigg|_{t=0} Ad_{\exp(tv)} = ad_v$$

(13)

On the other hand given any operator $D$, we can (aside from convergence issues) create the operator $\Exp(D)$ by

$$\Exp(D) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n$$

(14)

and note that $\frac{d}{dt} \Exp(tD) = \Exp(tD)D = D \Exp(tD)$. Because the solution to

$$\frac{d}{dt} M = M \circ ad_v$$

(15)

with initial conditions $M(0) = Id$ is unique, we have

$$Ad_{\exp(tv)} = \Exp(t ad_v).$$

(16)
The Canonical 1-form

3.1 $\mathfrak{g}$-valued $p$-forms

A Lie group, as a differentiable manifold, has a cotangent bundle and other associated bundles. A $\mathfrak{g}$-valued $p$-form $\alpha$ is simply a section of the bundle $\bigwedge^p \otimes \mathfrak{g}$ (where $\bigwedge^p$ is an abbreviation for $\bigwedge^p T^*M$). Letting $\{g_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$, a $\mathfrak{g}$-valued $p$-form $\alpha$ can be written

$$\alpha = \alpha^i \otimes g_i$$

$$= \alpha^i_{j_1 \ldots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p} \otimes g_i$$

where the $\alpha^i$ are $p$-forms and the $\alpha^i_{j_1 \ldots j_p}$ are functions.

Due to the bracket product on $\mathfrak{g}$, we can define a product on $(\bigoplus_{p=0}^n \bigwedge^p) \otimes \mathfrak{g}$. Letting $\alpha \in \bigwedge^p \otimes \mathfrak{g}$ and $\beta \in \bigwedge^q \otimes \mathfrak{g}$, we define

$$[\alpha, \beta] \in \bigwedge^{p+q} \otimes \mathfrak{g}$$

to be the $\mathfrak{g}$-valued $(p+q)$-form given by

$$[\alpha, \beta](v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in \sigma_{p+q}} (-1)^{\pi} \prod_{i=1}^{p+q} \alpha(v_{\pi i}, \ldots, v_{\pi p}), \beta(v_{\pi p+1}, \ldots, v_{\pi p+q})$$

where the sum is taken over all permutations $\pi$ in the symmetric group $\sigma_{p+q}$ on $p+q$ many symbols. Or, writing $\alpha = \alpha^i \otimes g_i$ and $\beta = \beta^j \otimes g_j$, we have

$$[\alpha, \beta] = \alpha^i \wedge \beta^j \otimes [g_i, g_j]$$

so the “bracket” on $\mathfrak{g}$-valued $p$-forms is a combination of exterior differentiation and the bracket operation. Sometime you will see symbols like $\alpha \wedge \beta$, which is not technically a section of $\bigwedge^{p+q} \otimes \mathfrak{g}$ but of $\bigwedge^{p+q} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

For instance, if $\alpha, \beta$ are $\mathfrak{g}$-valued 1-forms, we have

$$[\alpha, \beta](v, w) = [\alpha(v), \beta(w)] - [\alpha(w), \beta(v)]$$

Note in particular that

$$[\alpha, \alpha](v, w) = 2[\alpha(v), \alpha(w)].$$

Finally we have an “exterior differential”

$$d : \Gamma \left( \bigwedge^p \otimes \mathfrak{g} \right) \rightarrow \Gamma \left( \bigwedge^{p+1} \otimes \mathfrak{g} \right).$$
given on $\alpha = \alpha^i \otimes g_i \in (\Lambda^p \otimes g)$ by
\[ d(\alpha) = d\alpha^i \otimes g_i. \]  
(24)

Differential geometers will recognize this as covariant exterior differentiation on the flat bundle $g \to G \times g \to G$.

It is easily checked that the exterior derivative works well with the bracket operation:
\[ d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{|\alpha|}[\alpha, d\beta] \]  
(25)

where $\alpha, \beta$ are $g$-valued forms of order $|\alpha|$ and $|\beta|$.

### 3.2 The trivial, but not-so-trivial, canonical 1-form

Let $\theta$ be a the $g$-valued 1-form that assigns to a vector $v \in T_A G$ is associated element of $g$. In some sense, $\theta$ is an identity automorphism. It is common to write
\[ \theta_A = A^{-1}dA. \]  
(26)

This really is an abuse of the notation, but correctly indicates that tangent vectors at $A$ are brought back to vectors at $e$ via left-translation along $A^{-1}$. If $G$ is a group of matrices, then $A^{-1}dA$ has a precise meaning as written.

Because the Lie algebra bracket and the topological bracket are equivalent, note that
\[ \theta[v, w] = [v, w] = [\theta(v), \theta(w)]. \]  
(27)

Write $\theta = \theta^i \otimes g_i$. Letting $\{g_j\}_{j=1}^n$ be basis vectors extended to left-invariant fields, note that $\theta^i(g_j) = \delta_i^j$. Then
\[ d\theta(g_j, g_k) = (d\theta^i)(g_j, g_k) \cdot g_i \]
\[ = (g_j (\theta^i(g_k)) - g_k (\theta^i(g_j)) - \theta^i([g_j, g_k])) g_i \]
\[ = (g_j (\delta_i^j) - g_k (\delta_i^k) - \theta^i([g_j, g_k])) g_i \]
\[ = -\theta^i([g_j, g_k]) g_i \]
\[ = -[\theta(g_j), \theta(g_k)] \]
\[ = -\frac{1}{2} [\theta, \theta](g_j, g_k) \]  
(28)

Having checked it on basis vectors, we arrive at the Maurer-Cartan structure equation:
\[ d\theta + \frac{1}{2} [\theta, \theta] = 0. \]  
(29)