1 Review

1.1 Representations

Letting $\mathbb{R}^n$ have a positive definite inner product and using the notation $\Lambda^k = \wedge^k \mathbb{R}^n$ and $\Lambda$ for the exterior algebra, we have the twisted adjoint representation

$$\tilde{\text{Ad}} : Cl_n \longrightarrow O(n)$$

(1)
on $\Lambda^1$. We also have the representation

$$Cl_n \longrightarrow \text{Hom}(\Lambda, \Lambda)$$

(2)
generated by

$$v.\eta = v \wedge \eta - i_v \eta$$

(3)
where $v \in \mathbb{R}^n$ and $\eta \in \Lambda$. This representation passes to two kind of representations of $Cl_n^0$ on $\Lambda$, namely

$$Cl_n^0 \longrightarrow \text{Hom}(\Lambda^\text{even}, \Lambda^\text{even})$$

$$Cl_n^0 \longrightarrow \text{Hom}(\Lambda^\text{odd}, \Lambda^\text{odd})$$

(4)

$$Cl_n^0 \longrightarrow \text{Hom}(\Lambda^k, \Lambda^k)$$
The second, obtained simply by restriction of $\Lambda^k \to \Lambda^{k-2} \oplus \Lambda^k \oplus \Lambda^{k+2}$ to $\Lambda^k$, is simply the twisted adjoint representation of $\mathbb{R}^n$ extended in the standard way to $\Lambda^k \mathbb{R}^n$.

We obtain a $Spin(n) \subset Cl_n^0$ representation on $\Lambda^k$. This is a vector representation, and not strictly a spin representation. It passes to a derived representation of $spin(n) \approx so(n)$ on $\Lambda^k$.

In addition, we have a natural spin representation $\Delta$ for $Spin(n)$ when $n$ is odd, and two representations $\Delta^+, \Delta^-$ when $n$ is even. When $n \equiv 2 \text{ mod } 4$ we have a natural equivalence $\Delta^+ = \Delta^-$. When $n \equiv 4 \text{ mod } 4$ then $\Delta^+$ and $\Delta^-$ are inequivalent.

1.2 Dynkin Diagrams

In the case of $B_n$ we have the following fundamental $spin(2n+1)$ representations:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$\Lambda^1$};
\node (b) at (1,0) {$\Lambda^2$};
\node (c) at (2,0) {$\cdots$};
\node (d) at (3,0) {$\Lambda^{n-1}$};
\node (e) at (4,0) {$\Delta$};
\draw [->] (a) -- (b);
\draw [->] (b) -- (c);
\draw [->] (c) -- (d);
\draw [->] (d) -- (e);
\end{tikzpicture}
\end{center}

In the case of $D_n$ we have the following fundamental $spin(2n)$ representations:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$\Lambda^1$};
\node (b) at (1,0) {$\Lambda^2$};
\node (c) at (2,0) {$\cdots$};
\node (d) at (3,0) {$\Lambda^{n-3}$};
\node (e) at (4,0) {$\Lambda^{n-2}$};
\node (f) at (5,0) {$\Delta^+$};
\node (g) at (3,-1) {$\Delta^-$};
\draw [->] (a) -- (b);
\draw [->] (b) -- (c);
\draw [->] (c) -- (d);
\draw [->] (d) -- (e);
\draw [->] (e) -- (f);
\draw [->] (e) -- (g);
\end{tikzpicture}
\end{center}
1.3 Table of Spin Representations

We have

<table>
<thead>
<tr>
<th>n</th>
<th>$Cl_n$</th>
<th>$Cl_n^0$</th>
<th>Spin($n$)</th>
<th>Vec</th>
<th>$\triangle$</th>
<th>$\triangle^+$</th>
<th>$\triangle^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
<td>${\pm 1} \approx O(1)$</td>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R}$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{C}$</td>
<td>$\mathbb{R}$</td>
<td>$S^1 \mathbb{C}$</td>
<td>$SO(2)$</td>
<td>$\mathbb{R}^2$</td>
<td>$\mathbb{C} \oplus \overline{\mathbb{C}}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>2</td>
<td>$\mathbb{H} \oplus \mathbb{H}$</td>
<td>$\mathbb{H}$</td>
<td>$Sp(1)$</td>
<td>$\mathbb{H}^3$</td>
<td>$\mathbb{H}$</td>
<td>$\mathbb{H}^4$</td>
<td>$\mathbb{H}$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{H}(2)$</td>
<td>$\mathbb{H} \oplus \mathbb{H}^t$</td>
<td>$Sp(1) \oplus Sp(1)$</td>
<td>$\mathbb{R}^4$</td>
<td>$\mathbb{H} \oplus \mathbb{H}^t$</td>
<td>$\mathbb{H}$</td>
<td>$\mathbb{H}$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{C}(4)$</td>
<td>$\mathbb{H}(2)$</td>
<td>$Sp(2)$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{H}^8$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{R}^8$</td>
</tr>
<tr>
<td>5</td>
<td>$\mathbb{R}(8)$</td>
<td>$\mathbb{C}(4)$</td>
<td>$SU(4)$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{C}^4 \oplus \overline{\mathbb{C}}^4$</td>
<td>$\mathbb{C}^4$</td>
<td>$\overline{\mathbb{C}}^4$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{R}(8) \oplus \mathbb{R}(8)$</td>
<td>$\mathbb{R}(8)$</td>
<td>$Spin(7)$</td>
<td>$\mathbb{R}^7$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{R}^8$</td>
</tr>
<tr>
<td>7</td>
<td>$\mathbb{R}(16)$</td>
<td>$\mathbb{R}(8) \oplus \mathbb{R}(8)$</td>
<td>$Spin(8)$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{R}^8^+ \oplus \mathbb{R}^8^-$</td>
<td>$\mathbb{R}^8^+ \mathbb{R}^8^-$</td>
<td></td>
</tr>
</tbody>
</table>

The vector representation is simply the orthogonal $\tilde{Ad}$-representation of $Cl_n$ on $\mathbb{R}^n$, passed to Spin($n$), which acts by factoring through $SO(n)$. The $\triangle$ representation is the Spin($n$) representation obtained by restricting the basic $Cl_n$-representation. We have four basic cases:

$n \equiv 1 \text{ mod } 4$: Under Spin($4n+1$), the $Cl_{4n+1}$ representation splits into two equivalent representations, either one of which defines $\triangle$.

$n \equiv 2 \text{ mod } 4$: Under Spin($4n+2$), the $Cl_{4n+2}$ representation splits into two equivalent representations, which are naturally complex-conjugates of one another. We have

$$\triangle = \triangle^+ \oplus \triangle^- = \triangle^+ \oplus \overline{\triangle^+}.$$ 

$n \equiv 3 \text{ mod } 4$: Under Spin($4n+3$), either of the $Cl_{4n+3}$ representations is irreducible, and the two are equivalent; this defines $\triangle$.

$n \equiv 4 \text{ mod } 4$: The Clifford representation splits into two inequivalent representations under Spin($4n$).

We have

$$\triangle = \triangle^+ \oplus \triangle^-.$$ 

2 $A_n$ and Duality

A duality between real vector spaces $V, W$ is a bilinear map $F: V \otimes W \to \mathbb{R}$ so that given any non-zero $v \in V$ there is a $w \in W$ so that $F(v, w)$ is non-zero, and vice-versa. If $V$ and $W$ are normed vector space, we can define a normed duality, which is a duality $F$ so that $|F(v, w)| \leq \|v\|\|w\|$ and so that given any $v \in V$ there is a $w \in W$ for which equality is achieved, and vice-versa.
Dualities are common in mathematics; we know that if $V$ and $W$ have a duality, then $W = V^*$. If this duality is normed, we obtain a canonical isometric isomorphism $V \to V^*$.

Now consider the Lie algebras $\mathfrak{su}(n + 1)$. The complexifications are $\mathfrak{sl}(n, \mathbb{C})$, and the fundamental irreducible representations are the basic representation on $\Lambda^2 \mathbb{C} \approx \mathbb{C}$, which lifts to a unitary representation of $SU(n + 1)$. Similarly the $\mathfrak{sl}(n + 1, \mathbb{R})$ action on $\Lambda^i \mathbb{R}$ lifts to a $SL(2, \mathbb{R})$ action. By a process similar to that done above for the $B_n$ and $D_n$ algebras, one sees that the fundamental irreducible representations are $\Lambda^1, \Lambda^2, \ldots, \Lambda^{n-1}$.

We know there is a duality relation here: $(\Lambda^i)^* \approx \Lambda^{n+1-i}$, or that we have a natural map

$$F : \Lambda^i \otimes \Lambda^{n+1-i} \to \Lambda^{n+1} \approx \mathbb{F}$$

which we know is a normed duality in the case that $\mathbb{F} = \mathbb{R}$. This gives an explicit realization of the order-2 outer automorphism of the $A_n$ algebras:

The adjoint representation has highest weight $(1, 0, \ldots, 0, 1)$.

3 $D_4$ and Triality

3.1 Triality

A *triality* among three real vector spaces $V_1$, $V_2$, $V_3$ is a trilinear map

$$F : V_1 \otimes V_2 \otimes V_3 \to \mathbb{R}$$

so that for any non-zero $v_1 \in V_1$, $v_2 \in V_2$, there is a $v_3 \in V_3$ so that $F(v_1, v_2, v_3)$ is non-zero, and likewise for the other two cases. Note that if $v_1$ is fixed, then $F(v_1, \cdot, \cdot)$ is a duality between $V_2$ and $V_3$, and likewise for the other two cases. If $V_1$, $V_2$, $V_3$ are normed, a *normed triality* amongst them is a triality so that for any $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$ we have

$$|F(v_1, v_2, v_3)| \leq ||v_1|| ||v_2|| ||v_3||$$

and so that, given any $v_1 \in V_1$, $v_2 \in V_2$, there is some $v_3 \in V_3$ so that equality is obtained, and likewise for the other two cases.

Trialities are substantially less common than dualities, and normed trialities are very difficult to find. One way to find a normed triality is to build one from a product operation on a division algebra. Assuming we have a division algebra $\mathbb{D}$ and product map $\cdot$ with

$$\cdot : \mathbb{D} \otimes \mathbb{D} \to \mathbb{D}$$

4
Dualizing, we obtain

\[
\cdot : \mathbb{D} \otimes \mathbb{D} \otimes \mathbb{D}^* \rightarrow \mathbb{R}
\]  

(11)

We know that such algebras have an inner product, so we can dualize to obtain a product

\[
\cdot : \mathbb{D} \otimes \mathbb{D} \otimes \mathbb{D} \rightarrow \mathbb{R} \\
\cdot (A, B, C) = (A \cdot B, C).
\]  

(12)

To see that this is a normed triality, simply note

\[
| (A, B, C) | \leq \| A \cdot B \| \| C \| = \| A \| \| B \| \| C \|
\]  

(13)

and that equality is obtained for \( C = \pm A \cdot B \).

### 3.2 Trialities and Division Algebras

Now conversely, given a normed triality \( F \), we obtain a division algebra. Pick any unit vectors \( v_1 \in V_1 \), and \( v_2 \in V_2 \). Then there is a unit vector \( v_3 \in V_3 \) so that \( F(v_1, v_2, v_3) = 1 \). Since \( F(v_1, \cdot, \cdot) \) is a \( V_2 \), \( V_3 \) duality, we obtain an isometric isomorphism \( V_2 \leftrightarrow V_3 \) where \( v_2 \leftrightarrow v_3 \), and likewise \( V_1 \leftrightarrow V_3 \) and \( V_1 \leftrightarrow V_2 \), where \( v_1 \leftrightarrow v_2 \leftrightarrow v_3 \). Thus any triality \( F \) is in fact

\[
F : V \otimes V \otimes V \rightarrow \mathbb{R}
\]  

(14)

which is canonical only after choosing \( v_1, v_2 \). We have

\[
F : V \otimes V \rightarrow V^*
\]  

(15)

Now since \( F(v_1, \cdot, \cdot) \) is a duality, we have an isometric isomorphism \( V \rightarrow V^* \) so we obtain a new map which we'll call \( \star \):

\[
\star : V \otimes V \rightarrow V.
\]  

(16)

Note that \( v_1 \star b = b \) and \( a \star v_2 = a \), so we can identify \( v_1, v_2, v_3 \) with 1, the identity element.

Note that \( a \star b \) is an element \( c \in V \) so that \(|F(a, b, c)| = \| a \| \| b \| \| c \| \) so we have

\[
\| a \star b \| = \| a \| \| b \|
\]  

(17)

and we have obtained a division algebra.

### 3.3 \( D_4 \)

We have seen that if we can find a triality, we have a division algebra.
The diagram for $D_4$ is

\[
\begin{array}{c}
\Lambda^1 & \xrightarrow{\text{Adj.}} & \triangle^+ \\
\triangle^- & \xrightarrow{\text{Adj.}} & \Lambda^1
\end{array}
\]

This diagram has the largest symmetry group among all Dynkin diagrams; it is the symmetric group on 3 letters, and has a triality automorphism: an automorphism $A$ so that $A^3 = 1$. The diagram automorphisms correspond to the outer automorphisms of $\text{spin}(8)$, which interchange three of the four fundamental representations: $\Lambda^1$, $\triangle^+$ and $\triangle^-$, which therefore have the same dimension, but are inequivalent as we have already seen.

Now $\triangle^+$ and $\triangle^-$ are the $\pm 1$ eigenspaces of $\omega$, which, recall, commutes with $\text{spin}(8)$ but anticommutes with anything in $Cl^1_8$, including $\Lambda^1 \subset Cl^1_8$. Therefore we obtain a map

\[
\Lambda^1 \otimes \triangle^+ \rightarrow \triangle^-
\]

via Clifford multiplication. Because the action of $\Lambda^1$ is kernel-free, this produces a triality

\[
F : \Lambda^1 \otimes \triangle^+ \otimes \triangle^- \rightarrow \mathbb{R}
\]

Because the action of $\Lambda^1$ is isometric (orthogonal), this is a normed triality.

### 3.4 Automorphisms of the Triality map

The triality map above is $\text{Spin}(8)$-invariant. To see this, let $ab$ be any generator of $\text{Spin}(8)$, and recall its action on $\Lambda^1$ is the twisted adjoint action, and on $\triangle^\pm$ is left Clifford multiplication. Then if $v \in \Lambda^1$ and $s^+ \in \triangle^+$ is any left-handed spinor, we have

\[
(\tilde{A}_{ab}v) (ab s^+) = abvbaabs^+ = ab (vs^+).
\]

Notice that the unique irreducible representation of $\text{Spin}(7)$ is the spinor representation of dimension 8. Further, one can prove it is transitive on the unit sphere (maybe when there’s more time we’ll do this). Now there is a smaller $\text{Spin}(7)$ representation, the orthogonal representation on $\mathbb{R}^7$. After restricting to $\text{Spin}(7) \subset \text{Spin}(8)$, obtained by requiring a vector in $\Lambda^1 \mathbb{R}^8$ be fixed, we obtain representations of $\text{Spin}(7)$ on the spaces $\triangle^\pm$. To see that these are the irreducible 8-dimensional $\text{Spin}(7)$ representations, note that the Clifford
action of $Cl_8^0$ on $\Delta^\pm$ passes to $Cl_7$ under the standard isomorphism, so that $\Delta^\pm$ remain irreducible under $Cl_7$. If $\Delta^+$, say, were to reduce under $Cl_7^8$, it must split into $\mathbb{R}^7 \oplus \mathbb{R}$ where the representation on $\mathbb{R}^7$ is the orthogonal representation. However, this is impossible because $\wedge^1 \mathbb{R}^7 \subset Cl_7^1$ acts transitively on the unit spheres in both the 7- and 8-dimensional irreducible representations.

Thus $G_2$ is the subgroup of $Spin(7)$ that fixes a spinor, and

$$Spin(7)/G_2 = S^7.$$  \hspace{1cm} (22)