Lecture 2 - Lie Groups, Lie Algebras, and Geometry

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1 Overview

If \( D \) is any linear operator on a vector space, we can define \( \exp(D) \) by

\[
\exp(D) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n.
\]

(1)

The sum converges if the operator is bounded. In other cases, such as differential operators on Sobolev spaces, one has to deal with convergence on a case-by-case basis or work with densely defined operators. If \( A \) and \( B \) are commuting operators, we have

\[
\exp(A) \exp(B) = \exp(A + B).
\]

(2)

But the situation is bad in the non-commutative case. Expanding in terms of Taylor series shows that

\[
\exp(A) \exp(B) = \exp(A + B) + O(2)
\]

(3)

where \( O(2) \) consists of terms of quadratic and higher orders in \( A \) and \( B \). It is a simple matter to compute that in fact

\[
\exp(A) \exp(B) = \exp(A + B + \frac{1}{2}[A, B] + O(3))
\]

(4)

The Campbell-Baker-Hausdorff theorem asserts that \( O(3) \) can be written in terms of only bracket terms (things like \([A, [A, B]], [[B, [A, B]], B]\) and so forth) without terms like \( A^2 \) or \( ABA \).

2 Linear Lie Groups and Differential Geometry

A linear Lie group \( G \) is a Lie group whose underlying topological space is a set of differentiably varying matrices, with the group action being matrix multiplication. If \( Id \in G \) is the
identity, derivatives of paths through the identity can be identified with matrices, and the Lie derivative is simply the commutator of these matrices. In this way \( g = T_{ld} G \) is a Lie algebra of matrices.

The principle examples are the

\[
GL(n, F) = \left\{ M_{n \times n} \mid \det(M_{n \times n}) \neq 0 \right\}
\]

(5)

where \( F \) is the base field \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \) (there is no analogue for the octonions). Via realification, we need only really consider the case of base field \( \mathbb{R} \), although working with base fields \( \mathbb{R} \) or \( \mathbb{H} \) is conceptually and computationally simpler in many cases. The groups \( Gl(n, \mathbb{R}) \) are of course non-compact; however they have a large number of important subgroups:

\[
SL(n, \mathbb{R}) = \left\{ M \in GL(n, \mathbb{R}) \mid \det(M) = 1 \right\}
\]

(6)

which is still non-compact, but which has the compact subgroup

\[
SO(n, \mathbb{R}) = \left\{ M \in GL(n, \mathbb{R}) \mid M^T M = Id \right\}.
\]

(7)

A second compact subgroup, in the even dimensional case, is the symplectic group

\[
SP(2n, \mathbb{R}) = \left\{ M \in GL(n, \mathbb{R}) \mid M^T J M = J \right\}
\]

(8)

where

\[
J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}
\]

(9)

The \( Sp \) groups are related to both symplectic 2-forms and to unitary transformations of quaternionic vector spaces. Under the commutator bracket, we have the corresponding real Lie algebras

\[
\mathfrak{gl}(n, \mathbb{R}) = \text{all real } n \times n \text{ matrices}
\]

\[
\mathfrak{sl}(n, \mathbb{R}) = \left\{ x \in \mathfrak{gl}(n, \mathbb{R}) \mid tr(x) = 0 \right\}
\]

\[
\mathfrak{o}(n, \mathbb{R}) = \left\{ x \in \mathfrak{gl}(n, \mathbb{R}) \mid x^T + x = 0 \right\}
\]

\[
\mathfrak{sp}(2n, \mathbb{R}) = \left\{ x \in \mathfrak{gl}(2n, \mathbb{R}) \mid x^T J + J x = 0 \right\}
\]

(10)

The “realification” process alluded to above is the process of taking a complex \( n \times n \) matrix \( C = A + iB \) and writing it as real \( 2n \times 2n \) matrix

\[
C = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}
\]

(11)

and noting that the algebra laws carry over. However, if one wishes to work directly with complex matrices (as one often does), we have the groups

\[
GL(n, \mathbb{C}) = \text{Invertible } n \times n \text{ complex matrices}
\]

\[
SL(n, \mathbb{C}) = \left\{ M \in GL(n, \mathbb{C}) \mid \det(M) = 1 \right\}
\]

\[
SO(n, \mathbb{C}) = \left\{ M \in GL(n, \mathbb{C}) \mid M^T M = Id \right\}
\]

\[
SP(2n, \mathbb{C}) = \left\{ M \in GL(n, \mathbb{C}) \mid M^T J M = Id \right\}
\]

(12)
where $J$ is the same as before. In addition we have the very important unitary and special unitary groups

$$U(n) = \left\{ M \in GL(n, \mathbb{C}) \mid M^T M = 1 \right\}$$

$$SU(n) = U(n) \cap SL(n, \mathbb{C}).$$

We easily see that

$$\text{gl}(n, \mathbb{C}) = \text{gl}(n, \mathbb{R}) \otimes \mathbb{C}$$
$$\text{sl}(n, \mathbb{C}) = \text{sl}(n, \mathbb{R}) \otimes \mathbb{C}$$
$$\text{o}(n, \mathbb{C}) = \text{o}(n, \mathbb{R}) \otimes \mathbb{C}$$
$$\text{sp}(n, \mathbb{C}) = \text{sp}(n, \mathbb{R}) \otimes \mathbb{C}.$$ (14)

However, the unitary and special unitary algebras are real algebras

$$\text{u}(n) = \left\{ x \in \text{gl}(n, \mathbb{C}) \mid x^T + x = 0 \right\}$$
$$\text{su}(n) = \left\{ x \in \text{sl}(n, \mathbb{C}) \mid x^T + x = 0 \right\}.$$ (15)

as the condition $x^T + x = 0$ is not preserved by complex multiplication. However

$$\text{gl}(n, \mathbb{C}) = \text{u}(n) \otimes \mathbb{C}$$
$$\text{sl}(n, \mathbb{C}) = \text{su}(n) \otimes \mathbb{C}.$$ (16)

Many other special relations carry over; for instance unitary matrices are precisely the matrices that are both symplectic and orthogonal. If we define

$$Sp(n, \mathbb{H}) = \left\{ W \in GL(n, \mathbb{H}) \mid W^T W = 1 \right\}$$

then $Sp(n, \mathbb{H})$ is naturally embedded in $SP(2n, \mathbb{C})$ as a compact subgroup, and $Sp(2n, \mathbb{C})/Sp(n, \mathbb{H})$ is contractible (so $Sp(n, \mathbb{H})$ is a maximal compact subgroup).

2.1 Exponential maps

A linear Lie group $G$ has two exponential maps from $\mathfrak{g}$ to $G$, the first, denoted “exp” defined using the flow of the identity under left-invariant vector fields, and the other, denoted “Exp” defined using the exponential operator. These are easily seen to be the same operator. Note that for $v \in \mathfrak{g}$, we have $\frac{d}{dt}\big|_{t=0} Exp(tv) = v$, so at time 0 the paths $t \mapsto Exp(tv)$ and $t \mapsto exp(tv)$ share the same initial vector. Then note that the tangent vectors to both paths are invariant under push-forward by left-translation.

2.2 Riemannian Geometry

Let $G$ be a Lie group with Riemannian metric $g$. Obviously $g$ can be made left-invariant by placing a non-degenerate bilinear form $g$ on $\mathfrak{g}$ and then requiring $g_p(v, w) = g(L_{p^{-1}} v, L_{p^{-1}} w)$. 3
Therefore, with $g$ invariant under pullback along left-translations, the right-invariant fields are Killing fields.

Any compact Lie group has a bi-invariant metric. Such a metric is characterized by being both left-invariant, and having left-invariant fields as Killing fields. Thus

$$g([v, w], z) = g(v, [w, z])$$  \hspace{1cm} (18)

for left-invariant fields $v, w, \text{ and } z$ (aka “associativity”). In this case, the Koszul formula

$$2g(\nabla_x y, z) = ([x, y], z) - ([y, z], x) + ([z, x], y)$$ \hspace{1cm} (19)

$$= ([x, y], z)$$ \hspace{1cm} (20)

shows that

$$\nabla_x y = \frac{1}{2} [x, y].$$ \hspace{1cm} (21)

In particular the left-invariant fields integrate out to geodesics. Thus the exponential map from Lie group theory is the same as the exponential map of Riemannian geometry.

3 Examples

3.1 $SU(2)$

For certain reasons, this may be the most important example of a compact Lie group. Matrices $M \in \mathbb{C}(2)$ are unitary if $M^T M = Id$ and special if $det(M) = 1$. These are the matrices of the form

$$M = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \quad \text{where} \quad |z|^2 + |w|^2 = 1$$ \hspace{1cm} (22)

The standard topology gives this group the differentiable structure of $\mathbb{S}^3$. The Lie algebra $\mathfrak{su}(2)$ is the real span of the three trace-free antihermitian matrices

$$\bar{x}_1 = \sqrt{-1} \sigma_1 = \sqrt{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ \hspace{1cm} (23)

$$\bar{x}_2 = \sqrt{-1} \sigma_2 = \sqrt{-1} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$ \hspace{1cm} (24)

$$\bar{x}_3 = \sqrt{-1} \sigma_3 = \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ \hspace{1cm} (25)

which is the Lie algebra of purely imaginary quaternions under the commutator bracket, which is isomorphic to the cross product algebra.
Computing the adjoint maps on the Lie algebra $\text{su}(2)$, we find

\[
\begin{align*}
\text{ad} \vec{x}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \\
\text{ad} \vec{x}_2 &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \\
\text{ad} \vec{x}_3 &= \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{align*}
\] (26)

so the Killing form is the negative definite bilinear form

\[
\kappa = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}
\] (27)

and we put the bi-invariant Riemannian metric $g = -\frac{1}{8} \kappa$ on $SU(2)$, which gives the group constant curvature $+1$.

### 3.1.1 Hopf Fibration

The vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are represented by paths through the identity $\tau \mapsto \text{Exp}(\tau \vec{x}_i)$. Moving by left translation, at a point $M \in SU(2)$ the representative paths are $\tau \mapsto M \text{Exp}(\tau \vec{x}_i)$, which are the matrices $M \vec{x}_i$. Moving by right translation, at a point $M \in SU(2)$ the representative paths are $\tau \mapsto \text{Exp}(\tau \vec{x}_i) M$, which are the matrices $\vec{x}_i M$.

To get a feel for the difference between left- and right-invariant vector fields, consider the association between $SU(2)$ and the set $\mathbb{S}^3 \subset \mathbb{C}^2$:

\[
\begin{pmatrix} z \\ w \\ \bar{w} \end{pmatrix} \rightarrow (z, w) \in \mathbb{C}^2.
\] (28)

The left-invariant vector field from $\vec{x}_3$ gives rise to the diffeomorphisms by right-translation $\varphi_\tau(M) = R_{\text{Exp}(\tau \sqrt{-1} \sigma_3)} M = \text{Exp}(\tau \sqrt{-1} \sigma_3) M$, or

\[
\varphi_\tau \begin{pmatrix} z \\ w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} ze^{i\tau} \\ we^{i\tau} \\ \bar{w}e^{i\tau} \end{pmatrix} \quad \text{or} \quad \varphi_\tau(z, w) = (ze^{i\tau}, we^{i\tau})
\] (29)

so that flow lines of the left-invariant field $\vec{x}_3$ correspond to fibers of the Hopf map $(z, w) \mapsto zw^{-1}$.

The right-invariant vector field from $\vec{x}_3$ gives rise to the diffeomorphisms by left-translation $\varphi_\tau(M) = L_{\text{Exp}(\tau \sqrt{-1} \sigma_3)} M = \text{Exp}(\tau \sqrt{-1} \sigma_3) M$, or

\[
\varphi_\tau \begin{pmatrix} z \\ w \\ \bar{w} \end{pmatrix} = \begin{pmatrix} ze^{-i\tau} \\ we^{-i\tau} \\ \bar{w}e^{-i\tau} \end{pmatrix} \quad \text{or} \quad \varphi_\tau(z, w) = (ze^{-i\tau}, we^{-i\tau})
\] (30)
so that flow lines of the left-invariant field $\vec{x}_3$ correspond to fibers of the anti-Hopf map $(z, w) \mapsto zw^{-1}$.

### 3.1.2 $SU(2)$ as a spin group

Now consider an arbitrary trace-free anti-hermitian matrix, which can be interpreted as a vector in $\mathfrak{su}(2)$:

$$\vec{v} = a\vec{x}_1 + b\vec{x}_2 + c\vec{x}_3 = \begin{pmatrix} -ci \\ b - ci \\ -b - ci \end{pmatrix}. \tag{31}$$

Note that $\det(\vec{v}) = a^2 + b^2 + c^2 = |v|^2 = -\frac{1}{8} \kappa(v, v)$ is the norm-square of the vector. The metric is then

$$g(\vec{v}, \vec{w}) = \frac{1}{2} (\det(\vec{v} + \vec{w}) - \det(\vec{v}) - \det(\vec{w})). \tag{32}$$

In particular, if $M \in SU(2)$ (so in particular conjugation preserves), we have that $\text{Ad}_M : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is in fact orthogonal. Therefore

$$\text{Ad} : SU(2) \to SO(3). \tag{33}$$

Because $SU(2)$ is connected, the image is in a connected subgroup $O(3)$, so we have a Lie algebra epimorphism. The kernel of the $\text{Ad}$ map is easily seen to be $\pm \text{Id}$, giving a 2-1 covering map; indeed this is a universal covering map of $SO(3)$, as $SU(2)$ is simply-connected.

The double cover of a special orthogonal group $SO(n)$ is called its associated spinor group, denoted $Spin(n)$. We therefore have $Spin(3) = SU(2)$.

### 3.2 $SL(2, \mathbb{R})$

For certain reasons, $SL(2, \mathbb{R})$ may be the most important example of a non-compact Lie group. Considering its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \text{span}_\mathbb{R} \{ h, x, y \}$, we have

$$\text{ad}_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\text{ad}_x = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

$$\text{ad}_y = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$
so that
\[ \kappa = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \] (35)

In analogy with the \( SU(2) \) case, a good bi-invariant metric is
\[ g = -\frac{1}{8} \kappa \] (36)
which is Lorenzian, of signature \((+, -, -)\) (a time-like unit vector is \(x - y\), two space-like unit vectors are \(h\) and \(x + y\)). Setting \( \vec{v} = a\vec{h} + b\vec{x} + c\vec{y} \) again we compute
\[ \det(\vec{v}) = -a^2 - bc = -\frac{1}{8} \kappa(\vec{v}, \vec{v}) = ||\vec{v}||^2 \] (37)
so that
\[ g(\vec{v}, \vec{w}) = \frac{1}{2} (\det(\vec{v} + \vec{w}) - \det(\vec{v}) - \det(\vec{w})) . \] (38)

Because any other bi-invariant metric gives rise to an associative bilinear form on \( \mathfrak{sl}_2 \), and because any two associative forms are equal up to constant multiplication (which must be real if it comes from a metric), it follows that \( SL(2, \mathbb{R}) \) has no bi-invariant Riemannian metric.

Now consider the adjoint representation \( Ad : SL(2, \mathbb{R}) \rightarrow Hom(\mathfrak{sl}(2, \mathbb{R})) \). We have that conjugation with \( M \in SL(2, \mathbb{R}) \) preserves determinant, so therefore preserves the metric, giving us a map into \( O(1, 3) \), which is easily seen to be a surjection
\[ Ad : SL(2, \mathbb{R}) \rightarrow SO^+(1, 2) \] (39)
into the orthochronous special orthogonal group. The kernel is, again, seen to be \( \pm Id \), so this is a 2-1 covering map. Therefore
\[ SL(2, \mathbb{R}) \approx Spin(1, 2). \] (40)

Note however \( SL(2, \mathbb{R}) \) is not simply-connected. Its Lorenzian metric is Einsteinian with scalar curvature \(-2\), so its universal cover is a model of \((1 + 2)\)-dimensional anti-de-Sitter space (an empty universe with negative cosmological constant, or the Lorenzian analogue of hyperbolic space).

Of course \( SL(2, \mathbb{R}) \) has left-invariant Riemannian metrics; the natural geometry is of an \( S^1 \)-bundle over hyperbolic space, and is one of Thurston’s eight model geometries.

### 3.3 \( SL(2, \mathbb{C}) \)

We can prove the existence of a 2-1 map \( SL(2, \mathbb{C}) \rightarrow SO^+(1, 3) \) (also implying that \( SO^+(1, 3) \approx PSL(2, \mathbb{C}) \)). Start with any anti-Hermitian matrix
\[ \vec{x} = \begin{pmatrix} -i(t + z) \\ -y - ix \\ y - ix \\ -i(t - z) \end{pmatrix} = t \vec{x}_0 + x \vec{x}_1 + y \vec{x}_2 + z \vec{x}_3 \] (41)
(under the commutator bracket, this is the Lie algebra of $U(2)$, which is reductive but not semi-simple). Then

$$-\det \vec{x} = t^2 - x^2 - y^2 - z^2$$

is the Lorenzian norm-square. The Lorenzian inner product is therefore

$$(\vec{x}, \vec{y}) = -\frac{1}{2} (\det(\vec{x} + \vec{y}) - \det(\vec{x}) - \det(\vec{y})).$$

Now $SL(2, \mathbb{C})$ acts on the set of anti-hermitian matrices via the conjugation isomorphism

$$Conj : SL(2, \mathbb{C}) \to SO^+(1, 3), \quad Conj_M(\vec{x}) = M\vec{x}M^T.$$  

This is clearly norm-preserving, although $Conj_M$ is generally not an algebra homomorphism.

It is easy to see that $ker(Conj) = \{ \pm Id \}$, so this is again a 2-1 covering map. Further, $SL(2, \mathbb{C})$ is simply-connected, so

$$SL(2, \mathbb{C}) \cong Spin(1, 3)$$

and we have found another spin group.