Lecture 3 - Cambpell-Baker-Hausdorff

January 18, 2013

Reference for this lecture: Lie Algebras, 2004, by Shlomo Sternberg

1 Statement

Set

$$\psi(z) \;=\; \frac{z}{z-1}\log(z) \tag{1}$$

and note that this is continuous at $z = 1$. In fact we have the taylor series

$$\psi(1+z) \;=\; \frac{1+z}{z}\log(1+z) \;=\; \frac{1+z}{z}\left(z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots\right) \tag{2}$$

$$= \; 1 - \sum_{n=1}^{\infty}(-1)^n \frac{1}{n(n+1)} z^n \tag{3}$$

The Campbell-Baker-Hausdorff formula, which we will prove in the case of Lie groups, is

$$exp(A)\, exp(B) \;= \exp\left(A + \int_0^1 \psi\left((Exp\, ad_A)\,(Exp\, t\, ad_B)\right) B\; dt\right) \tag{4}$$

for non-commuting operators A, B when the appropriate sums converge.

2 Proof of CBH

2.1 Initial Considerations

Let G be a Lie group and let $C(t)$ be any differentiable path in $\mathfrak{g}$. Let $g : \mathbb{R}^2 \to G$ be the function

$$g(s,t) \;=\; exp(s\, C(t)). \tag{5}$$

Consider the pullback $g^*\theta$ to $\mathbb{R}^2$, written

$$g^*\theta \;=\; \alpha(s,t)\,ds \;+\; \beta(s,t)\,dt \tag{6}$$

where α, β are $\mathfrak{g}$-valued functions. A path representing $\frac{\partial}{\partial s}$ at the point $g(s,t) \in G$ is $\tau \mapsto exp((s+\tau)C(t)) = exp(\tau C(t))exp(sC(t))$. Therefore

$$\begin{aligned}
\alpha \;&=\; \theta\left(\frac{\partial}{\partial s}\right) \\
&=\; L_{exp(-sC(t))*}\frac{d}{d\tau}\Big|_{\tau=0} exp(\tau C(t))exp(sC(t)) \\
&=\; \frac{d}{d\tau}\Big|_{\tau=0} exp(-sC(t))exp(\tau C(t))exp(sC(t)) \\
&=\; C(t)
\end{aligned} \tag{7}$$

but that

$$\beta \;=\; L_{exp(-sC(t))*}\frac{d}{d\tau}\Big|_{\tau=t} exp(sC(\tau)) \tag{8}$$

is more difficult to evaluate, due to the non-commutativity of $C'(t)$ and $C(t)$. However the Maurer-Cartan equation gives

$$\begin{aligned}
\left(\frac{\partial\beta}{\partial s} \;-\; \frac{\partial\alpha}{\partial t}\right) \;+\; [\alpha,\,\beta] \;&=\; 0 \\
\frac{\partial\beta}{\partial s} \;&=\; C'(t) \;-\; [C(t),\,\beta]\,.
\end{aligned} \tag{9}$$

Writing

$$\beta(s,t) \;=\; a_0 \;+\; a_1 s \;+\; a_2 s^2 \;+\; \ldots \tag{10}$$

where $a_i = a_i(t)$ are $\mathfrak{g}$-valued functions of t. We have $a_0 = 0$ because $\beta(0,t) = 0$ by (8). Plugging in to (9) we get

$$a_1 \;+\; \sum_{n=1}^{\infty}(n+1)a_{n+1}s^n \;=\; C'(t) \;-\; \sum_{n=1}^{\infty}[C(t),\,a_n]s^n \tag{11}$$

giving us the recursive equations

$$a_1 \;=\; C'(t) \qquad a_n \;=\; -\frac{1}{n}\,ad_C(t)\,a_{n-1} \tag{12}$$

so that

$$\begin{aligned}
a_n \;&=\; \frac{1}{n!}\left(-ad_{C(t)}\right)^{n-1} C'(t) \\
\beta(s,t) \;&=\; \left(\sum_{n=1}^{\infty}\frac{1}{n!}\left(-s\,ad_{C(t)}\right)^{n-1}\right) sC'(t)
\end{aligned} \tag{13}$$

The expression on the right contains the Taylor series for $\varphi(z) = \frac{e^z - 1}{z}$, so

$$\beta(s,\, t) \quad = \quad \phi(-s\, ad_{C(t)})\, sC'(t) \tag{14}$$

Setting $s = 1$ we have an expression for the lgrithmic derivative of an exponential:

$$L_{exp(-C(t))*} \frac{d}{d\tau}\Big|_{\tau=t} exp(C(\tau)) \;=\; \phi(-ad_{C(t)})\, C'(t) \tag{15}$$

2.2 Main Proof

Set

$$\Gamma(t) \;=\; \log\left(exp(A)\, exp(tB)\right) \tag{16}$$

so that Γ is a path in $\mathfrak{g}$, for sufficiently small A, B and for $|t| < 1$. We have

$$\begin{aligned} exp\, \Gamma(t) &= exp(A) exp(tB) \\ Ad_{exp\, \Gamma(t)} &= Ad_{exp(A)} Ad_{exp(tB)} \\ Exp\, ad_{\Gamma(t)} &= Exp(ad_A)\, Exp(t\, ad_B) \\ ad_{\Gamma(t)} &= Log\left(Exp(ad_A)\, Exp(t\, ad_B)\right). \end{aligned} \tag{17}$$

Pick t and consider the path $\tau \mapsto \exp \Gamma(t+\tau)$, which represents a vector at $\exp \Gamma(t)$. This corresponds to the Lie algebra element

$$\begin{aligned} \theta\left(\frac{d}{d\tau}\right) &= L_{\exp(\Gamma(s))^{-1}*} \frac{d}{d\tau} \exp \Gamma(t+\tau) \\ &= \frac{d}{d\tau}\left((\exp \Gamma(t))^{-1} \exp \Gamma(t+\tau)\right) \\ &= \frac{d}{d\tau} \exp(-tB) \exp(\tau B) \exp(tB) \\ &= B \end{aligned} \tag{18}$$

On the other hand from (15) we have

$$L_{\exp(\Gamma(t))^{-1}*} \frac{d}{d\tau} \exp \Gamma(t+\tau) \;=\; \phi(-ad_{\Gamma(t)})\, \Gamma'(t). \tag{19}$$

Therefore

$$\begin{aligned} \Gamma'(t) &= \left[\phi(-ad_{\Gamma(t)})\right]^{-1} B \\ &= \left[\phi(-Log(Exp(ad_A) Exp(t\, ad_B)))\right]^{-1} B \\ &= \psi(Exp(ad_A) Exp(t\, ad_B))\, B \end{aligned} \tag{20}$$

Since $\Gamma(0) = A$ and $\Gamma(1) = log\left(exp(A) exp(B)\right)$, we have our formula

$$exp(A)\, exp(B) \;=\; exp\left(A \,+\, \int_0^1 \psi(Exp(ad_A) Exp(t\, ad_B))\, B\, dt\right). \tag{21}$$

3 Application: Derived Representations

Given a group representation of G on a vector space V, an associated representation of $\mathfrak{g}$ on V exists. If $g \in \mathfrak{g}$, it is represented by a path $\gamma_g : (-\epsilon, \epsilon) \to G$ through the identity; we can set $\gamma_g(\tau) = exp(\tau g)$. If $m \in V$, define $g.m$ by

$$g.m \;=\; \frac{d}{d\tau}\gamma_g(\tau).m \tag{22}$$

To verify this is a representation we must check that $[g,h].m = g.h.m - h.g.m$. To do so we use the CBH formula. First

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0}\frac{d}{d\tau}\Big|_{\tau=0} e^{-sg}e^{-\tau h}e^{sg}e^{\tau h}.m \;&=\; \frac{d}{ds}h.m \;+\; \frac{d}{ds}\frac{d}{d\tau}e^{-sg}e^{-\tau h}e^{sg}.m \\
&=\; \frac{d}{d\tau}\frac{d}{ds}e^{-sg}e^{-\tau h}e^{sg}.m \\
&=\; \frac{d}{d\tau}e^{-\tau h}.g.m \;+\; \frac{d}{d\tau}\frac{d}{ds}e^{-sg}e^{-\tau h}.m \\
&=\; -h.g.m \;+\; \frac{d}{ds}\frac{d}{d\tau}e^{-sg}e^{-\tau h}.m \\
&=\; -h.g.m \;-\; \frac{d}{ds}e^{-sg}.h.m \\
&=\; -h.g.m \;+\; g.h.m
\end{aligned} \tag{23}$$

Next, using the Campbell-Baker-Hausdorff formula,

$$\begin{aligned}
e^{sg}e^{\tau h} \;&=\; exp\left(sg \;+\; \tau h \;+\; \frac{1}{2}s\tau[g,h] \;+\; \frac{1}{12}s^2\tau[g,[g,h]] \;+\; \frac{1}{12}s\tau^2[h,[h,g]] \;+\; O(4)\right) \\
e^{-sg}e^{-\tau h} \;&=\; exp\left(-sg \;-\; \tau h \;+\; \frac{1}{2}s\tau[g,h] \;-\; \frac{1}{12}s^2\tau[g,[g,h]] \;-\; \frac{1}{12}s\tau^2[h,[h,g]] \;+\; O(4)\right) \\
e^{-sg}e^{-\tau h}e^{sg}e^{\tau h} \;&=\; exp\left(s\tau[g,h] \;+\; O(3)\right)
\end{aligned} \tag{24}$$

Therefore

$$\begin{aligned}
\frac{d}{ds}\frac{d}{d\tau}e^{-sg}e^{-\tau h}e^{sg}e^{\tau h}.m \;&=\; \frac{d}{ds}\frac{d}{d\tau}exp\left(s\tau[g,h] \;+\; O(3)\right).m \\
&=\; \frac{d}{ds}\left(s[g,h] \;+\; O(2)\right).m \\
&=\; [g,h].m
\end{aligned} \tag{25}$$

so that $[g,h].m \;=\; (g.h. - h.g.)m$. The derived representation is therefore indeed a Lie algebra representation.