Lecture 3 - Campbell-Baker-Hausdorff

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Reference for this lecture: Lie Algebras, 2004, by Shlomo Sternberg

1 Statement

Set
\[
\psi(z) = \frac{z}{z-1} \log(z)
\]
and note that this is continuous at \( z = 1 \). In fact we have the taylor series
\[
\psi(1 + z) = 1 + \frac{1 + z}{z} \log(1 + z) = \frac{1 + z}{z} \left( z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \ldots \right)
\]
\[
= 1 - \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n+1)} z^n
\]
The Campbell-Baker-Hausdorff formula, which we will prove in the case of Lie groups, is
\[
\exp(A) \exp(B) = \exp \left( A + \int_0^1 \psi(\text{Exp} \left( \text{ad}_A \right) \left( \text{Exp} \left( \text{ad}_B \right) \right) B \ dt \right)
\]
for non-commuting operators \( A, B \) when the appropriate sums converge.

2 Proof of CBH

2.1 Initial Considerations

Let \( G \) be a Lie group and let \( C(t) \) be any differentiable path in \( g \). Let \( g : \mathbb{R}^2 \to G \) be the function
\[
g(s, t) = \exp(s \ C(t)).
\]
Consider the pullback $g^* \theta$ to $\mathbb{R}^2$, written
\begin{equation}
 g^* \theta = \alpha(s,t) \, ds + \beta(s,t) \, dt \tag{6}
\end{equation}
where $\alpha, \beta$ are $\mathfrak{g}$-valued functions. A path representing $\frac{\partial}{\partial s}$ at the point $g(s,t) \in G$ is $\tau \mapsto \exp((s+\tau)c(t)) = \exp(\tau C(t)) \exp(s C(t))$. Therefore
\begin{align*}
 \alpha &= \theta \left( \frac{\partial}{\partial s} \right) \\
 &= \left. L_{\exp(-sC(t))} \frac{d}{d\tau} \right|_{\tau=0} \exp(\tau C(t)) \exp(s C(t)) \\
 &= \left. \frac{d}{d\tau} \right|_{\tau=0} \exp(-sC(t)) \exp(\tau C(t)) \exp(s C(t)) \\
 &= C(t) \tag{7}
\end{align*}
but that
\begin{equation}
 \beta = L_{\exp(-sC(t))} \left. \frac{d}{d\tau} \right|_{\tau=t} \exp(s C(t)) \tag{8}
\end{equation}
is more difficult to evaluate, due to the non-commutativity of $C'(t)$ and $C(t)$. However the Maurer-Cartan equation gives
\begin{equation}
 \left( \frac{\partial \beta}{\partial s} - \frac{\partial \alpha}{\partial t} \right) + [\alpha, \beta] = 0 \tag{9}
\end{equation}
Writing
\begin{equation}
 \beta(s,t) = a_0 + a_1 s + a_2 s^2 + \ldots \tag{10}
\end{equation}
where $a_i = a_i(t)$ are $\mathfrak{g}$-valued functions of $t$. We have $a_0 = 0$ because $\beta(0,t) = 0$ by (8). Plugging in to (9) we get
\begin{equation}
 a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}s^n = C'(t) - \sum_{n=1}^{\infty} [C(t), a_n]s^n \tag{11}
\end{equation}
giving us the recursive equations
\begin{equation}
 a_1 = C'(t) \quad a_n = -\frac{1}{n} \, ad_{C(t)} a_{n-1} \tag{12}
\end{equation}
so that
\begin{equation}
 a_n = \frac{1}{n!} \left( -ad_{C(t)} \right)^{n-1} C'(t) \quad \beta(s,t) = \left( \sum_{n=1}^{\infty} \frac{1}{n!} \left( -s \, ad_{C(t)} \right)^{n-1} \right) s C'(t) \tag{13}
\end{equation}
The expression on the right contains the Taylor series for $\varphi(z) = \frac{e^z - 1}{z}$, so

$$\beta(s, t) = \phi(-s \text{ad}_{C(t)}) s C'(t) \quad (14)$$

Setting $s = 1$ we have an expression for the logarithmic derivative of an exponential:

$$L_{\exp(-C(t))} \left. \frac{d}{dt} \exp(C(t)) \right|_{t=t} = \phi(-\text{ad}_{C(t)}) C'(t) \quad (15)$$

2.2 Main Proof

Set

$$\Gamma(t) = \log (\exp(A) \exp(tB)) \quad (16)$$

so that $\Gamma$ is a path in $g$, for sufficiently small $A$, $B$ and for $|t| < 1$. We have

$$\begin{align*}
\exp \Gamma(t) &= \exp(A) \exp(tB) \\
\text{Ad}_{\exp \Gamma(t)} &= \text{Ad}_{\exp(A)} \text{Ad}_{\exp(tB)} \\
\exp \text{ad}_{\Gamma(t)} &= \exp(\text{ad}_A \exp(t \text{ad}_B)) \\
\text{ad}_{\Gamma(t)} &= \log(\exp(\text{ad}_A \exp(t \text{ad}_B))).
\end{align*}$$

(17)

Pick $t$ and consider the path $\tau \mapsto \exp \Gamma(t + \tau)$, which represents a vector at $\exp \Gamma(t)$. This corresponds to the Lie algebra element

$$\theta \left( \frac{d}{dt} \right) = L_{\exp(\Gamma(t))} \left. \frac{d}{dt} \exp(t + \tau) \right|_{t=t}$$

$$= \frac{d}{dt} \left( (\exp(t + \tau))^{-1} \exp(t + \tau) \right)$$

$$= \frac{d}{dt} \exp(-tB) \exp(\tau B) \exp(tB) = B \quad (18)$$

On the other hand from (15) we have

$$L_{\exp(\Gamma(t))} \left. \frac{d}{dt} \exp(t + \tau) \right|_{t=t} = \phi(-\text{ad}_{\Gamma(t)}) \Gamma'(t). \quad (19)$$

Therefore

$$\Gamma'(t) = \left[ \phi(-\text{ad}_{\Gamma(t)}) \right]^{-1} B$$

$$= \left[ \phi(-\text{Log}(\exp(\text{ad}_A \exp(t \text{ad}_B)))) \right]^{-1} B$$

$$= \psi(\exp(\text{ad}_A \exp(t \text{ad}_B))) B \quad (20)$$

Since $\Gamma(0) = A$ and $\Gamma(1) = \log(\exp(A) \exp(B))$, we have our formula

$$\exp(A) \exp(B) = \exp \left( A + \int_0^1 \psi(\exp(\text{ad}_A \exp(t \text{ad}_B))) B \, dt \right). \quad (21)$$

3
3 Application: Derived Representations

Given a group representation of $G$ on a vector space $V$, an associated representation of $g$ on $V$ exists. If $g \in g$, it is represented by a path $\gamma_g : (-\epsilon, \epsilon) \to G$ through the identity; we can set $\gamma_g(\tau) = \exp(\tau g)$. If $m \in V$, define $g.m$ by

$$ g.m = \frac{d}{d\tau} \gamma_g(\tau).m $$

(22)

To verify this is a representation we must check that $[g,h].m = g.h.m - h.g.m$. To do so we use the CBH formula. First

$$ \frac{d}{ds} \left|_{s=0} \frac{d}{d\tau} \right|_{\tau=0} e^{-sg}e^{-\tau h}e^{sg}e^{\tau h}.m = \frac{d}{ds} h.m + \frac{d}{ds} \frac{d}{d\tau} e^{-sg}e^{-\tau h}e^{sg}.m $n

$$ = \frac{d}{d\tau} e^{-sg}e^{-\tau h}e^{sg}.m $$

$$ = \frac{d}{d\tau} e^{-\tau h}.g.m + \frac{d}{d\tau} \frac{d}{ds} e^{-sg}e^{-\tau h}.m $$

$$ = -h.g.m + \frac{d}{d\tau} \frac{d}{ds} e^{-sg}e^{-\tau h}.m $$

$$ = -h.g.m - \frac{d}{ds} e^{-sg}.h.m $$

$$ = -h.g.m + g.h.m $$

(23)

Next, using the Campbell-Baker-Hausdorff formula,

$$ e^{sg}e^{\tau h} = \exp \left( sg + \tau h + \frac{1}{2} s^2 \tau [g,h] + \frac{1}{12} s^3 \tau^2 [g,[g,h]] + \frac{1}{12} s^3 \tau^2 [h,[h,g]] + O(4) \right) $$

$$ e^{-sg}e^{-\tau h} = \exp \left( -sg - \tau h + \frac{1}{2} s^2 \tau [g,h] - \frac{1}{12} s^3 \tau^2 [g,[g,h]] - \frac{1}{12} s^3 \tau^2 [h,[h,g]] + O(4) \right) $$

$$ e^{-sg}e^{-\tau h}e^{sg}e^{\tau h} = \exp \left( s\tau [g,h] + O(3) \right) $$

(24)

Therefore

$$ \frac{d}{ds} \frac{d}{d\tau} e^{-sg}e^{-\tau h}e^{sg}e^{\tau h}.m = \frac{d}{ds} \frac{d}{d\tau} \exp \left( s\tau [g,h] + O(3) \right).m $$

$$ = \frac{d}{ds} \left( s[g,h] + O(2) \right).m $$

$$ = [g,h].m $$

(25)

so that $[g,h].m = (g.h - h.g) m$. The derived representation is therefore indeed a Lie algebra representation.