Lecture 9 - Representation Theory I: Examples

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1 Finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ modules: review

A consequence of the semisimplicity of $\mathfrak{sl}(2,\mathbb{C})$ is that any element $x \in \mathfrak{sl}(2,\mathbb{C})$ can be written $x = x_s + x_n$, where x_s, x_n have the property that $[x_s, x_n] = 0$, and under any representation $\psi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$, we have that $\psi(x_s)$, is a semisimple linear operator and $\psi(x_n)$ is a nilpotent linear operator. You will recall that this was a consequence of $Der(\mathfrak{g}) \approx \mathfrak{g}$ (under the adjoint representation), which is the statement that $H^1(\mathfrak{g},\mathfrak{g}) = 0$. The decomposition $x = x_s + x_n$ is called the abstract jordan decomposition of $x \in \mathfrak{g}$.

A maximal toral subalgebra (a subalgebra maximal under the condition that all elements are semisimple) is necessarily abelian and self-centralizing. In the case of $\mathfrak{sl}(2,\mathbb{C}) \approx$ *span*$\{h, x, y\}$ we have that h is abstractly semisimple and x, y are abstractly nilpotent. We have

$$\begin{aligned} ad_h\, x &= 2x \\ ad_h\, y &= -2y \end{aligned} \tag{1}$$

so we call x the positive and y the negative root.

If $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ is an irreducible finite dimensional representation, there is some vector $v_0 \in V$ so that $x.v = 0$. This is called a highest vector. Defining $v_i = \frac{1}{i!} y^i.v_0$, there exists some $\lambda \in \mathbb{N}$ so that

a) $h.v_i \;=\; (\lambda - 2i)\, v_i$

b) $x.v_i \;=\; (\lambda - i + 1)\, v_{i-1}$

c) $y.v_i \;=\; (i+1)\, v_{i+1}$

Then h is called the *weight operator*, a vector v is homogenous of weight α if $h.v = \alpha v$, and x, y are called the *creation* and *destruction* operators, respectively, and act by raising or lowering weights, respectively. The space $span_{\mathbb{C}}\{v_0, \ldots, v_n\}$ is called a weight-string. The weights in a weight-string are always of the form $\lambda, \lambda - 2, \ldots, -\lambda$.

Thus finite-dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ consist of a single weight string, and the weight spaces are all one-dimensional. We shall denote the irreducible module of highest weight Λ by V^λ, and the weight space of weight m by V^λ_m.

Finally we have the Casimir operator $c \in Z(U(\mathfrak{sl}(2, \mathbb{C})))$. On the module of highest weight Λ we have $c = \frac{1}{2}\lambda\,(\lambda + 2)$.

2 Physical interpretation

The Laws of physics generally have Poincare-invariance, meaning invariance under the natural representation of the non-compact group $SO(3) \ltimes \mathbb{R}^3$. In the case that forces act centrally, this can be reduced to an $SO(3)$-invariance, which then gives a derived $\mathfrak{so}(3)$ action. In the case of quantum mechanics, the state of a system is given by a vector $\phi \in \mathcal{H}$, where $\mathcal{H}$ is complex Hilbert space (the vector space of complex-valued L^2 functions on $\mathbb{R}^3$), so we naturally have an action of $\mathfrak{so}(3) \otimes \mathbb{C} \approx \mathfrak{sl}(2, \mathbb{C})$. If X, Y, Z are the natural $\mathfrak{so}(3)$ generators (which act naturally by partial differentiation), the explicit isomorphism is

$$\begin{aligned} l_z &= 2\,i\,Z \\ l_+ &= -Y + iX = i\,(X + iY) \\ l_- &= Y + iX = i\,(X - iY) \end{aligned} \tag{2}$$

Restricting to spherical shells, say the shell of radius 1, we retain the representation, and the Casimir operator is

$$c = -\triangle_{\mathbb{S}^2} \tag{3}$$

which we now know takes on the values $\frac{1}{2}\lambda(\lambda + 2)$ for integral λ.

A function $\varphi : \mathbb{S}^2 \to \mathbb{C}$ is said to have *pure A-state* if it is an A-eigenvector. For example, φ has pure X-angular momentum of magnitude α if $X.\varphi = \alpha\varphi$. Vectors cannot obtain a pure state in two non-commuting variables simultaneously, but any vector can be described (up to linear combinations) by specifying simultaneous values for c and l_z (which *do* commute). These are, respectively, the "azimuthal" quantum number, normally given by $l = \frac{1}{2}\lambda$, and the "magnetic" quantum number m, where $m \in \{-l, -l + 1, \ldots, l\}$. For $SO(3)$ representations, λ is even, so both quantum numbers are integral.

Assuming a vector $\psi \in \mathcal{H}$ is simultaneously in both a pure c- and l_z-state, we interpret c as its *total energy operator* (in this case, the total angular momentum operator), and l_z as the quantity of its angular momentum about the z-axis. The operators l_+ and l_- are nilpotent: the operator l_+ has the effect of creating one quantum of right-handed angular momentum about the z-axis in exchange for one quantum of left-handed angular momentum. The "destruction" operator l_- does the opposite. This suggests the appropriate way to view the weight-string is horizontally (left-to-right) and not vertically, and to view l_+ and l_- as exchange, not creation/destruction, operators.

3 Tensor products and the fundamental representation

Given two representations V and W of any Lie algebra $\mathfrak{g}$, the spaces $V \oplus W$ and $V \otimes W$ have natural structure as Lie algebra represenations. Specifically,

$$\begin{aligned} g.\,(v, w) &= (g.v,\, g.w) \\ g.\,(v \otimes w) &= g.v \otimes w \,+\, v \otimes g.w \end{aligned} \tag{4}$$

Consider the simplest non-trivial $\mathfrak{sl}(2, \mathbb{C})$ representation, $V = V_1$, the 2-dimensional representation. Let

$$V \;=\; V^1 \;=\; span_{\mathbb{C}}\{z,\, w\} \tag{5}$$

where

$$\begin{aligned} h.z &= z \quad & h.w &= -w \\ x.z &= 0 \quad & x.w &= z \\ y.z &= w \quad & y.w &= 0 \end{aligned} \tag{6}$$

Now let $V^{\otimes 2}$ be $V \otimes V$ with its natural representation structure. We have the natural $\mathfrak{sl}_2$-module decomposition

$$\begin{aligned} V^{\otimes 2} &= span_{\mathbb{C}}\left\{zz,\, \frac{1}{2}(zw + wz),\, ww\right\} \oplus span_{\mathbb{C}}\left\{\frac{1}{2}(zw - wz)\right\} \\ &= V^2 \oplus V^0 \end{aligned} \tag{7}$$

so that $V \otimes V$ is naturally the direct sum of the representations V^2 and V^0.

In any case, if V and W are arbitrary representations with highest weights λ_v, λ_w, then $V \otimes W$ will be a representation (not necessarily irreducible) with highest weight (quite evidently) being $\lambda_v + \lambda_w$. Letting $V = V^1$ again, we have that $V^{\otimes k}$ has highest weight k. Thus by taking tensor products of V with itself, we can obtain any (finite-dimensional) $\mathfrak{sl}(2, \mathbb{C})$ representation whatsoever. We therefore call V^1 the *fundamental representation*.

Note that the group $SL(2, \mathbb{C})$ also has a natural action on V_1, via ordinary matrix multiplication. It has a natural action on tensor products coming from

$$A\,(v_1 \otimes v_2) \;=\; Av_1 \,\otimes\, Av_2, \tag{8}$$

which then provides the natural action of $SL(2, \mathbb{C})$ on all of the V^λ.

4 Automorphisms of $\mathbb{P}^n$

Reference: Fulton-Harris, Representation Theory, Ch 13

Modulo the constants, the degree-1 polynomials in z, w is just $\mathbb{P}^1$, which then has an action of $SL(2,\mathbb{C})$ that factors through $PSL(2,\mathbb{C})$ (via the kernel $\pm I \in SL(2,\mathbb{C})$, so is a 2 to 1 covering map).

Above we identified the irreducible $\mathfrak{sl}_2$-module of weight n as $V^n = S^n V^1$. One counts basis elements and sees that $Dim(S^n V^1) = n+1$, so that

$$\mathbb{P}(S^n V^1) \; = \; \mathbb{P}^n \tag{9}$$

which then inherits an $SL(2,\mathbb{C})$-action (which, again, naturally factors through $PSL(2,\mathbb{C})$). Of course $SL(2,\mathbb{C})$ is too small to be transitive, and in any case we know the automorphism group is $PSL(n+1,\mathbb{C})$ (and has symmetry group $PSU(n+1,\mathbb{C})$, when given the usual homogeneous metric). The question is, what are the $SL(2,\mathbb{C})$-orbits?

Now consider the $\mathbb{P}^1 \to \mathbb{P}^n$ Veronese embedding $\iota^n : \mathbb{P}V^1 \to \mathbb{P}(S^n V^1)$, given by the projectivization of the homogeneous map $\iota^n(v) = v^{\odot n}$. On the one hand, the image of $\mathbb{P}^1$ is clearly preserved by the $SL(2,\mathbb{C})$-action. On the other hand given a point $[\alpha, \beta] \in \mathbb{P}^1$, which is represented by the vector $\alpha z + \beta w$, we have

$$[\alpha,\, \beta] \; = \; \left[\alpha^n,\, n\alpha^{n-1}\beta,\, \ldots,\, \beta^n\right] \tag{10}$$

which is represented by the vector $\alpha^n z^n + n\alpha^{n-1}\beta z^n w \cdots + \beta^n w^n \in V^{\odot(n+1)}$, making it easy to see that any element of $SL(n+1,\mathbb{C})$ which fixes all such points is the identity. Therefore this $PSL(2,\mathbb{C})$ action is precisely the group of automorphisms of $\mathbb{P}^n$ that preserves the image $\iota^n(\mathbb{P}^1)$.