Lecture 1 - Statement

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The aim of this series of lectures is to apply methods of analysis to problems in Riemannian geometry.

1 Best Metrics

1.1 Motivations

A motivating problem is the elucidation of the set optimal metrics on a given differentiable manifold. Even what "optimal" means is a matter of contention, but normally includes "most symmetrical possible," given the underlying topology. For instance round metrics on $\mathbb{S}^2$ are by nearly any consideration "optimal'— in particular it has maximum possible symmetry, as it is homogeneous and isotropic. Note that no Killing fields exist on compact surfaces of negative Euler characteristic, so global symmetries do not exist (a result of Bochner's); likewise for Einstein metrics on manifolds of negative einstein constant. However, the constant curvature metrics are invariant on the universal cover.

"Optimal" metrics are often found as minimizers of a global "energy" functionals, to be discussed below. Does a given differential manifold possess a metric that optimizes some functional? What do minimizing sequences look like? If no such metric exists, how do minimizing sequences degenerate? Can we hope for any kind of "geometric decomposition" akin to Thurston-Perelman geometrization of 3-manifolds?

A second fundamental problem, possibly more tractible, is the question of how many optimal metrics a manifold may possess. If, for instance, a manifold M admits an Einstein metric, how many einstein metrics might it possess? Do these metrics collect to form any kind of domain, or are they "isolated"?

1.2 Moduli Spaces

The *space of Riemannian metrics*, $Met(M)$, on a differentiable manifold M is the set of symmetric 2-tensors g on M so that g is positive definite. Clearly this is a cone. The *moduli space of Riemannian metrics* on M is the space of metrics, modulo the relation of diffeomorphism:

$$Mod(M) \;=\; Met(M) \;/\; Diff(M). \tag{1}$$

In the positiive case, the situation is rigid: the manifolds $\mathbb{S}^2$ and $\mathbb{R}P^2$ each admit precisely one metric of sectional curvature $+1$, up to diffeomorphism. Notice that there are many *metrics* of constant section al curvature on $\mathbb{S}^2$. For instance, if φ is any fractional linear transformation, then it is a diffeomorphism on $\mathbb{S}^2$ (indeed in the complex setting, it is a biholomorphism). Since the fractional linear transformations are just $SL(2,\mathbb{R})$ as a group, and because this group is non-compact, there are divergent sequences of metrics of constant curvature $+1$ on $\mathbb{S}^2$. Working with moduli spaces rather than metrics simplifies matters considerably—in essence it filters away "bad" coordinate choices.

1.3 The L^2 metric on $Met(M)$ and $Mod(M)$

Among possible metric structures on the moduli space of Riemannian metrics is the L^2-distance—among it's many disadvantages it has the advantage of being easy to define. Given a metric g on a compact manifold M, any symmetric 2-tensor h can be regarded as a "tangent vector," in the sense that the family of 2-tensors $s \mapsto g + sh$ is a metric for sufficiently small values of s. We can define the "length" of the vector h at the metric g to be

$$\left(\int_M h_{ij}h_{kl}g^{ik}g^{jl}\,dVol_g\right)^{\frac{1}{2}} \;=\; \left(\int_M |h|_g^2\,dVol_g\right)^{\frac{1}{2}} \;=\; \|h_{ij}\|_{L^2,g} \tag{2}$$

Then given an arbitrary continuously differentiable family of metrics $g(s)$, $s \in [0,1]$ we have the length of the path

$$\left(\int_{t=0}^{1} \left\|\frac{dg}{ds}\right\|_{L^2,g(s)}^2 dt\right)^{\frac{1}{2}} \tag{3}$$

Whether a minimizing path exists between two metrics is not certain, but at the very least, in the collection of paths from one metric to another the L^2 path length has an infimum. This is the L^2 distance—it is invariant under $Diff(M)$, so it passes to $Mod(M)$. Later we shall discuss other distance functions, such as the Gromov-Hausdorff distance.

2 Canonical metrics

One typical definition (not necessarily universally agreed upon) for the term "canonical metric" is a metric that optimizes a Riemannian curvature functional. The Euler-Lagrange equation for a well-chosen functional is normally some form of elliptic equation for some geometric quantity or quantities. This provides a way for analytic techniques to be brought to bear on the problem of understanding these metrics.

Common functionals are the quadratic curvature functionals

$$\int |\,\mathrm{Rm}\,|^2 \, dVol, \qquad \int R^2 \, dVol, \qquad \int |W|^2 \, dVol \tag{4}$$

and the Einstein-Hilbert functional

$$\int R \, dVol. \tag{5}$$

The Euler-Lagrange equations of each of these are relatively tractible; in particular, if the variation is $g_{ij}(s) = g_{ij} + s h_{ij}$, a straightforward computation provides

$$\frac{d}{ds} \int R \, dVol \;=\; - \int \left\langle \mathrm{Ric} \,-\, \frac{1}{2} R \, g, \, h \right\rangle_g dVol_g \tag{6}$$

so that the Euler-Lagrange equations are

$$\mathrm{Ric}_{ij} \,-\, \frac{1}{2} R \, g_{ij} \;=\; 0. \tag{7}$$

Except in dimension 2, this implies that the metric is Ricci-flat. In particular $\int R \, dVol = 0$ is the only critival value. In harmonic coordiates, we have the quasi-linear 2^{nd} order system of equations

$$\triangle \, (g_{ij}) \,+\, Q_{ij}(g, \partial g) \;=\; -2\mathrm{Ric}_{ij} \tag{8}$$

so the Euler-Lagrange equations (that is, $\mathrm{Ric}_{ij} = 0$) do indeed provide an elliptic condition.

However, consider the homogeneity of these functionals. As distance is scaled by d, the n-form $dVol$ is scales by d^n, $|\,\mathrm{Rm}\,|^2$, etc scale by d^{-4}, and R scales by d^{-2}. Thus the quadratic functionals can be expected only to have non-trivial minimizers only in dimension 4, and the Einstein-Hilbert functional has non-trivial minimizers only in dimension 2. To see what is meant by this, if we replace g by $d^2 g$, we obtain

$$\int_M R_{d^2 g} \, dVol_{d^2 g} \;=\; d^{n-2} \int_M R_g \, dVol_g \tag{9}$$

so letting $d \searrow 0$ minimizes this functional if it is positive, and $d \to \infty$ minimizes it if it is negative somewhere.

A word about the other quadratic functionals in the scale-invariant 4-dimensional case: optimizers of $\int |\operatorname{Rm}|^2$ (as metric-compatible connections are varied) are metrics of harmonic curvature, namely $\operatorname{Rm}_{ijk}{}^{l}{}_{,l} = 0$, which is a first-order elliptic equaiton. Opimizers of $\int |W|^2$ are *Bach-flat*, which is an elliptic condition, provided sectional curvature is specified (to be constant, for instance).

Possibly we could consider scale-invariant functionals such as

$$\int |\operatorname{Rm}|^{\frac{n}{2}}\, dVol, \tag{10}$$

but the Euler-Lagrange equations are essentially un-interpretable. It is better to consider scale-invariant functionals of the type

$$\operatorname{Vol}^{\frac{4-n}{n}} \int |\operatorname{Rm}|^2\, dVol \qquad or \qquad \operatorname{Vol}^{\frac{2-n}{n}} \int R\, dVol. \tag{11}$$

With regards the Einstein-Hilbert functional, we find that

$$\begin{aligned} \frac{d}{ds}\left(\operatorname{Vol}^{\frac{2-n}{n}} \int R\, dVol\right) &= \left(\frac{2-n}{2n}\right) \operatorname{Vol}^{\frac{2-n}{n}-1} \int \langle g,\, h\rangle\, dVol \int R\, dVol \\ &\quad - \operatorname{Vol}^{\frac{2-n}{n}} \int \left\langle \operatorname{Ric} - \frac{1}{2}Rg,\, h\right\rangle dVol. \end{aligned} \tag{12}$$

Letting

$$\Lambda = \frac{n-2}{2n} \operatorname{Vol}^{-1} \int R\, dVol \tag{13}$$

be a multiple of the average scalar curvature, we have

$$\frac{d}{ds}\left(\operatorname{Vol}^{\frac{2-n}{n}} \int R\, dVol\right) = -\operatorname{Vol}^{\frac{2-n}{n}} \int \left\langle \operatorname{Ric} - \frac{1}{2}Rg + \Lambda\, g,\, h\right\rangle dVol. \tag{14}$$

so the Euler-Lagrange equations are therefore

$$\operatorname{Ric}_{ij} - \frac{1}{2}Rg_{ij} + \Lambda\, g_{ij} = 0. \tag{15}$$

We still have

$$\frac{n}{2}R - R = n\Lambda \tag{16}$$

so that R is constant (when $n \neq 2$), but not necessarily zero. Since $\frac{2n}{n-2}\Lambda = R$, the Euler-Lagrange equation can be rewritten

$$\overset{\circ}{\operatorname{Ric}}_{ij} = 0. \tag{17}$$

where

$$\overset{\circ}{\operatorname{Ric}}_{ij} \triangleq \operatorname{Ric}_{ij} - \frac{1}{n}Rg_{ij} \tag{18}$$

is the *trace-free Ricci tensor*. Such metrics are called *Einstein metrics*.

3 Einstein metrics

Throughout these lectures we shall emphasize the Einstein case. Indeed Einstein metrics are the best understood of the canonical metrics, and the theory is the most complete.

In the 2-dimensional case, the situation is very well understood. The equation

$$\mathrm{Ric} \;=\; \frac{1}{2}Rg \tag{19}$$

is vacant. However if $R = const$ is imposed, this is the equation of constant sectional curvature. After scaling, we may assume $R \in \{+1, 0, -1\}$. The Gauss-Bonnet formula

$$\chi(\Sigma) \;=\; \frac{1}{2\pi}\int R\,dVol \tag{20}$$

imposes volume control on such manifolds (when $\chi \neq 0$). In the case $R = -1$, it is impossible that injectivity radii are *uniformly* small.

The stuation here is very well understood: every closed Riemannian 2-manifold admits a metric of sectional curvature $+1$, 0, or -1, as its Euler number is greater than zero, zero, or less than zero, respectively. As we have seen, the sphere, and therefore the real projective plane, have trivial moduli spaces. The Torus and Klein bottles each admit 2-parameter moduli spaces. A surface Σ of negative Euler number, however, admits families of metrics of sectional curvature -1 (metrics of constant negative sectional curvature are in 1-1 correspondance with complex structures on such a manifold; to learn more, consult the theory of uniformization of surfaces). Further, the moduli spaces can be non-compact, and "cusp" singularities can form in the limit.

The 3-dimensional case is likewise tractible. Einstein metrics are Ricci-constant, and therefore have constant sectional curvature. In addition, in the negative case, moduli are trivial. The question of which manifolds admit such metrics has been answered elsewhere.

The 4-dimensional case is currently at the center of a great deal of activity. It is well-known that the Riemann tensor splits into four orthogonal pieces:

$$\mathrm{Rm} \;=\; \frac{R}{24}g\circ g \,+\, \frac{1}{2}\overset{\circ}{\mathrm{Ric}}\circ g \,+\, W^+ \,+\, W^- \tag{21}$$

where $\circ$ is used to denote the Kulkarni-Nomizu product. The tensors W^+ and W^- are orthogonal components of the Weyl tensor. They are characterized as follows. Recall that the Hodge star operator $*$ is an idempotent on $\bigwedge^2$, so that is eigenvalues are ± 1. Therefore

$$\bigwedge^2 \;=\; \bigwedge^+ \oplus \bigwedge^-. \tag{22}$$

Because $*$ is an orthogonal map, we have that $\left\langle \bigwedge^+, \bigwedge^- \right\rangle = 0$. If Rm and W are regarded as linear maps $\bigwedge^2 \to \bigwedge^2$, one can show

$$*W \;=\; W* \tag{23}$$

so that W preserves the $\bigwedge^+$ and $\bigwedge^-$ eigenspaces. We simply set $W^+ = W\big|_{\bigwedge^+}$ and $W^- = W|_{\bigwedge^-}$.

The Chern-Gauss-Bonnet formulae in dimension 4 read

$$\begin{aligned}\chi(M) &= \frac{1}{8\pi^2}\int \frac{1}{24}R^2 - \frac{1}{2}|\overset{\circ}{\mathrm{Ric}}|^2 + |W|^2 \\ \tau(M) &= \frac{1}{12\pi^2}\int |W^+|^2 - |W^-|^2\end{aligned} \tag{24}$$

provide a great deal of information. For instance, the L^2 norm of the Riemann tensor is

$$|\,\mathrm{Rm}\,|^2 = \frac{1}{6}R^2 + 2|\overset{\circ}{\mathrm{Ric}}|^2 + |W|^2. \tag{25}$$

In the case of compact Einstein 4-manifolds, $L^2(|\,\mathrm{Rm}\,|)$ is uniformly bounded by $\chi(M)$! In addition,

$$\begin{aligned}2\chi(M) + 3\tau(M) &= \frac{1}{4\pi^2}\int \frac{1}{24}R^2 + 2|W^+|^2 \geq 0 \\ 2\chi(M) - 3\tau(M) &= \frac{1}{4\pi^2}\int \frac{1}{24}R^2 + 2|W^-|^2 \geq 0.\end{aligned} \tag{26}$$

The inequality $2\chi \pm 3\tau \geq 0$ is known as the *Hitchin-Thorpe inequality* (in addition, equality occurs only when M is flat, or a quotient of a Calabi-Yau manifold, as shown by Hitchin).

4 Euler-Lagrange Equations

The Euler-Lagrange equations for the scale-invariant Einstein-Hilbert functional are precisely that

$$\overset{\circ}{\mathrm{Ric}} = 0. \tag{27}$$

This is in fact an elliptic condition. One way to see this is by using the Binchi identities to compute $\triangle\,\mathrm{Rm}$. We have

$$\begin{aligned}\mathrm{Rm}_{ijkl,ss} &= -\mathrm{Rm}_{ijls,ks} - \mathrm{Rm}_{ijsk,ls} \\ &= -\mathrm{Rm}_{ijls,sk} - \mathrm{Rm}_{ksti}\mathrm{Rm}_{tjls} - \mathrm{Rm}_{kstj}\mathrm{Rm}_{itls} - \mathrm{Rm}_{kstl}\mathrm{Rm}_{ijts} - \mathrm{Rm}_{ksts}\mathrm{Rm}_{ijlt} \\ &\quad - \mathrm{Rm}_{ijsk,sl} - \mathrm{Rm}_{lsti}\mathrm{Rm}_{tjsk} - \mathrm{Rm}_{lstj}\mathrm{Rm}_{itsk} - \mathrm{Rm}_{lsts}\mathrm{Rm}_{ijtk} - \mathrm{Rm}_{lstk}\mathrm{Rm}_{ijst}\end{aligned} \tag{28}$$

$$\begin{aligned}\mathrm{Rm}_{ijkl,ss} &= \mathrm{Rm}_{sils,jk} + \mathrm{Rm}_{jsls,ik} + \mathrm{Rm}_{sisk,jl} + \mathrm{Rm}_{jssk,il} \\ &\quad - \mathrm{Rm}_{ksti}\mathrm{Rm}_{tjls} - \mathrm{Rm}_{kstj}\mathrm{Rm}_{itls} - \mathrm{Rm}_{kstl}\mathrm{Rm}_{ijts} - \mathrm{Rm}_{ksts}\mathrm{Rm}_{ijlt} \\ &\quad - \mathrm{Rm}_{lsti}\mathrm{Rm}_{tjsk} - \mathrm{Rm}_{lstj}\mathrm{Rm}_{itsk} - \mathrm{Rm}_{lsts}\mathrm{Rm}_{ijtk} - \mathrm{Rm}_{lstk}\mathrm{Rm}_{ijst}\end{aligned} \tag{29}$$

$$\begin{aligned}\mathrm{Rm}_{ijkl,ss} &= \mathrm{Ric}_{il,jk} - \mathrm{Rm}_{jl,ik} - \mathrm{Rm}_{ik,jl} + \mathrm{Rm}_{jk,il} \\ &\quad - \mathrm{Rm}_{ksti}\mathrm{Rm}_{tjls} - \mathrm{Rm}_{kstj}\mathrm{Rm}_{itls} - \mathrm{Rm}_{kstl}\mathrm{Rm}_{ijts} - \mathrm{Rm}_{ksts}\mathrm{Rm}_{ijlt} \\ &\quad - \mathrm{Rm}_{lsti}\mathrm{Rm}_{tjsk} - \mathrm{Rm}_{lstj}\mathrm{Rm}_{itsk} - \mathrm{Rm}_{lsts}\mathrm{Rm}_{ijtk} - \mathrm{Rm}_{lstk}\mathrm{Rm}_{ijst}\end{aligned} \tag{30}$$

Schematically, we write

$$\triangle \operatorname{Rm} \;=\; \nabla^2 \operatorname{Ric} + \operatorname{Rm} * \operatorname{Rm}\,. \tag{31}$$

Now consider

$$\begin{aligned} \frac{1}{2}\triangle|\operatorname{Rm}|^2 \;&=\; |\operatorname{Rm}|\triangle|\operatorname{Rm}| \;+\; |\nabla|\operatorname{Rm}||^2 \\ \frac{1}{2}\triangle|\operatorname{Rm}|^2 \;&=\; \langle \operatorname{Rm},\, \triangle \operatorname{Rm}\rangle \;+\; |\nabla \operatorname{Rm}|^2 \end{aligned} \tag{32}$$

and that $|\nabla|\operatorname{Rm}||^2 \;\leq\; |\nabla \operatorname{Rm}|^2$. We have

$$\begin{aligned} |\operatorname{Rm}|\triangle|\operatorname{Rm}| \;&=\; \langle \operatorname{Rm},\, \triangle \operatorname{Rm}\rangle \;+\; |\nabla \operatorname{Rm}|^2 \;-\; |\nabla|\operatorname{Rm}||^2 \\ &\geq\; \langle \operatorname{Rm},\, \operatorname{Rm} * \operatorname{Rm}\rangle \\ &\geq\; -C|\operatorname{Rm}|^3. \end{aligned} \tag{33}$$

where $C = C(n)$. Setting $u = \frac{1}{C}|\operatorname{Rm}|$, then off the zero-locus we have

$$\triangle u \;\geq\; -u^2 \tag{34}$$

On the zero locus, this equation continues to hold in the barrier sense.

5 The Role of Analysis

Standard references on the theory of second order elliptic differential equations discuss the regularity of inequalities of the form

$$\triangle u \;\geq\; -f\,u \tag{35}$$

where $f \geq 0$. Encouragingly, we find the following typical result:

Theorem 5.1 *If $\Omega \subset \mathbb{R}^n$ is a pre-compact domain, if $u \in L^1(\Omega)$, if*

$$\triangle u \;\geq\; -fu, \quad f \geq 0 \tag{36}$$

and if $f \in L^p$ for some $p > \frac{n}{2}$, then

$$u \in L^\infty \tag{37}$$

where $\|u\|_{L^\infty}$ has a uniform bound depending on n, p, and $\|f\|_{L^p}$.

In our equation (34), we can set $f = Cu$, and so analysis holds out the possibility that some L^p bound on $u = |\operatorname{Rm}|$ implies *uniform pointwise sectional curvature bounds.*

Now the minimization of a functional might itself provide such a bound, but one notices that the scale-invariant functionals we have considered can only provide bounds at the critical value $p = \frac{n}{2}$, or else on $p = 2$ which is of little help when $n \geq 4$. Indeed the 4-dimensional case is the critical case for this reason. Analysis does indeed give sectional curvature bounds a.e., along with the phenomenon of "concentration of curvature." We shall explore the details and implications in the following lectures.

6 The Sobolev Inequality: the nexus of geometry and analysis

If Ω is an n-dimensional domain with a Riemannian metric and $\nu > 0$, we define the ν-isoperimetric constant of Ω to be

$$I_\nu(\Omega) \;=\; \sup_{\Omega' \subset\subset \Omega} \frac{\operatorname{Vol}(\Omega')^{\frac{\nu-1}{\nu}}}{\operatorname{Area}(\partial\Omega')}$$

where Area indicates Hausdorff $(n-1)$-measure. If Ω is a closed Riemannian manifold, we take the infimum over domains Ω' with $\operatorname{Vol}\Omega' \leq \frac{1}{2}\operatorname{Vol}\Omega$; if some such restriction is not made then of course the infimum is zero. Note that if $\nu < n$ then $I_\nu(\Omega) = \infty$.

On the other hand we define the ν-Sobolev constant of Ω by

$$S_\nu(\Omega) \;=\; \sup_{f \in C_c^\infty(\Omega)} \frac{\left(\int_\Omega |f|^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}}{\int_\Omega |\nabla f|}.$$

If Ω is a closed Riemannian manifold, we take the infimum over functions with $\operatorname{Vol}(\operatorname{supp} f) < \frac{1}{2}\operatorname{Vol}(\Omega)$; if some such restriction is not made then of course the infimum is zero.

Theorem 6.1 (Federer-Fleming)

$$I_\nu(\Omega) \;=\; S_\nu(\Omega).$$

_Pf_

Pf that $S_\nu(\Omega) \leq I_\nu(\Omega)$.

With

$$\int |\nabla f| \;\geq\; S_\nu(\Omega)^{-1}\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}},$$

we can let $f \equiv 1$ on Ω', $f \equiv 0$ outside ${\Omega'}^{(\epsilon)}$ (the ϵ-thickening of Ω'), and $f(p) = 1 - \epsilon^{-1}\operatorname{dist}(\Omega', p)$ on ${\Omega'}^{(\epsilon)} - \Omega'$. As $\epsilon \searrow 0$ we have

$$\lim_{\epsilon\searrow 0}\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \;=\; \operatorname{Vol}(\Omega')^{\frac{\nu-1}{\nu}}$$

$$\lim_{\epsilon\searrow 0}\int |\nabla f| \;=\; \lim_{\epsilon\searrow 0}\frac{\operatorname{Vol}({\Omega'}^{(\epsilon)} - \Omega')}{\epsilon} \;=\; \operatorname{Area}(\partial\Omega').$$

Therefore

$$\operatorname{Area}(\partial\Omega') \;=\; \lim_{\epsilon\searrow 0}\int |\nabla f| \;\geq\; \lim_{\epsilon\searrow 0} S_\nu(\Omega)^{-1}\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \;=\; S_\nu(\Omega)^{-1}\operatorname{Vol}(\Omega')^{\frac{\nu-1}{\nu}}.$$

Pf that $I_\nu(\Omega) \le S_\nu(\Omega)$.

To simplify the proof, assume that f is C^∞ and critical points of f are isolated; any $C^{0,1}$ function is a limit such functions. Given a nonnegative C_c^∞ function $f : \Omega \to \mathbb{R}$ and given a number t, let $A_t = f^{-1}(t)$ and let $\Omega_t = f^{-1}([t, \infty])$. Locally (near a regular point of f) we can parametrize Ω' by letting f be one coordinate, and putting some coordinates on A_t. We can split the cotangent bundle by letting $df/|df|$ be one covector in an orthonormal coframe. Then if $d\sigma_t$ indicates the wedge product of the remaining vectors, we have Then $dV = \frac{1}{|\nabla f|} df \wedge d\sigma_t$. Therefore

$$\begin{aligned} \int_M |\nabla f|\, dV &= \int_{min(f)}^{max(f)} \int_{A_t} d\sigma_t\, df \;=\; \int_0^\infty \mathrm{Area}(A_t)\, dt \\ &\ge I_\nu(\Omega)^{-1} \int_0^\infty \mathrm{Vol}(\Omega_t)^{\frac{\nu-1}{\nu}}\, dt \end{aligned}$$

The equality $\int_M |\nabla f|\, dV \;=\; \int_0^\infty \mathrm{Area}(A_t)\, dt$ is called the *coarea formula*. Changing the order of integration, *á la* calculus III, gives

$$\begin{aligned} \int f^{\frac{\nu}{\nu-1}} &= \frac{\nu}{\nu-1} \int_\Omega \int_0^{f(p)} t^{\frac{1}{\nu-1}}\, dt\, dVol(p) \\ &= \frac{\nu}{\nu-1} \int_0^\infty \int_{\Omega_t} t^{\frac{1}{\nu-1}}\, dV\, dt \;=\; \frac{\nu}{\nu-1} \int_0^\infty t^{\frac{1}{\nu-1}}\, \mathrm{Vol}(\Omega_t)\, dt \end{aligned}$$

The result follows from the following lemma.

Lemma 6.2 *If $g(t)$ is a nonnegative decreasing function and $s \ge 1$, then*

$$\left(s \int_0^\infty t^{s-1} g(t)\, dt \right)^{\frac{1}{s}} \;\le\; \int_0^\infty g(t)^{\frac{1}{s}}\, dt$$

Pf

We have

$$\begin{aligned} \frac{d}{dT} \left(s \int_0^T t^{s-1} g(t)\, dt \right)^{\frac{1}{s}} &= T^{s-1} g(T) \left(s \int_0^T t^{s-1} g(t)\, dt \right)^{\frac{1}{s}-1} \\ &\le T^{s-1} g(T)^{\frac{1}{s}} \left(s \int_0^T t^{s-1}\, dt \right)^{\frac{1}{s}-1} \;=\; g(T)^{\frac{1}{s}}. \end{aligned}$$

Since $\frac{d}{dT} \int_0^T g(t)^{\frac{1}{s}}\, dt \;=\; g(T)^{\frac{1}{s}}$, we have

$$\left(s \int_0^T t^{s-1} g(t)\, dt \right)^{\frac{1}{s}} \;\le\; \int_0^T g(t)^{\frac{1}{s}}\, dt$$

for all T. $\square$

7 Suggested Problems

1) Let $g(t) = g + sh$ for $s \in (-\epsilon, \epsilon)$ be a family of metrics, and let $x^1, \ldots, x^n$ be a coordinate system.

   a) Define $h^{ij} = h_{st}g^{si}g^{tj}$, and let $dVol$ be the alternatig n-tensor $dVol = \sqrt{det\, g_{ij}}dx^1 \wedge \cdots \wedge dx^n$. Verify the following variation formulae for the following tensors quantities:

$$\begin{aligned} \frac{d}{ds}g^{ij} &= -h^{ij} \\ \frac{d}{ds}\log\, det\, g &= \langle h,\, g\rangle \\ \frac{d}{ds}dVol &= \frac{1}{2}\langle h,\, g\rangle\, dVol. \end{aligned} \tag{38}$$

   b) Verify the following variation formulae for the Christoffel symbols:

$$\frac{d}{ds}\Gamma_{ij}^k = \frac{1}{2}\left(\nabla_{\partial/\partial x^i}\, h_{js} + \nabla_{\partial/\partial x^i}\, h_{is} - \nabla_{\partial/\partial x^s}\, h_{ij}\right) g^{sk} \tag{39}$$

   c) Verify the following variation formulae for the Ricci and scalar curvatures:

$$\begin{aligned} \frac{d}{ds}\mathrm{Ric}_{ij} &= \frac{1}{2}\left(\mathrm{Rm}_{ijtj}{}^m h_{ms} + h_{is,jt} + h_{jt,si}\right) g^{st} \\ &\quad + \frac{1}{2}\left(\mathrm{Ric}_i{}^m h_{mj} - (Tr\, h)_{,ij} - (\triangle h)_{ij}\right) \\ \frac{d}{ds}R &= -\triangle(Tr\, h) + h_{su,tv}g^{st}g^{uv} + \langle \mathrm{Ric},\, h\rangle_g \end{aligned} \tag{40}$$

   (hint: use the expression for Ric_{ij} in Christoffel symbols, and assume coordinates are chosen so $\Gamma_{ij}^k = 0$ at a point p).

2) Using the set-up and results from probelm (1), verify explicitly that

$$\begin{aligned} \frac{d}{ds}\,\mathrm{Vol}\,(M) &= \frac{1}{2}\int \langle h,\, g\rangle_g\, dVol_g \\ \frac{d}{ds}\int R\, dVol &= -\int \left\langle \mathrm{Ric} - \frac{1}{2}Rg,\, h\right\rangle dVol \\ \frac{d}{ds}\left(\mathrm{Vol}(M)^{\frac{2-n}{n}}\int_M R\, dVol\right) &= -\mathrm{Vol}(M)^{\frac{2-n}{n}}\int_M \left\langle \mathrm{Ric} - \frac{1}{2}Rg + \Lambda g,\, h\right\rangle dVol \end{aligned} \tag{41}$$

3) If Ω is any domain in a Riemannian manifold, show that $I_\nu(\Omega) = \infty$ when $\nu < n$ (hint: show this first for domains in $\mathbb{R}^n$).

4) If Ω is not pre-compact, show, by example, that $S_\nu(\Omega) = \infty$ unless $\nu = n$.

5) Develop a *geodesic equation* for paths $g(s)$ in the L^2 metric on $Met(M)$. This is done, in the usual way, by creating a variation $g_t(s)$ of the path $g(s)$ and requiring that the L^2-length be stable (have t-derivative zero) under an arbitrary such variation (in the C^1 category).