Lecture 2 - Geometric Convergence

Brian Weber

July 2, 2013

1 The Role of Analysis

Recall that on Einstein n-manifolds, we have

$$\triangle \operatorname{Rm} = \operatorname{Rm} * \operatorname{Rm}. \tag{1}$$

Now we have

$$\begin{aligned} \frac{1}{2}\triangle|\operatorname{Rm}|^2 &= |\operatorname{Rm}|\triangle|\operatorname{Rm}| + |\nabla|\operatorname{Rm}||^2 \\ \frac{1}{2}\triangle|\operatorname{Rm}|^2 &= \langle \operatorname{Rm}, \triangle \operatorname{Rm}\rangle + |\nabla \operatorname{Rm}|^2 \end{aligned} \tag{2}$$

and $|\nabla|\operatorname{Rm}||^2 \le |\nabla \operatorname{Rm}|^2$. Therefore

$$\begin{aligned} |\operatorname{Rm}|\triangle|\operatorname{Rm}| &= \langle \operatorname{Rm}, \triangle \operatorname{Rm}\rangle + |\nabla \operatorname{Rm}|^2 - |\nabla|\operatorname{Rm}||^2 \\ &\ge \langle \operatorname{Rm}, \operatorname{Rm} * \operatorname{Rm}\rangle \\ &\ge -C|\operatorname{Rm}|^3. \end{aligned} \tag{3}$$

where $C = C(n)$. Setting $u = |\operatorname{Rm}|$, then off the zero-locus we have

$$\triangle u \ge -Cu^2 \tag{4}$$

On the zero locus, this equation continues to hold in the barrier sense.

Standard references on the theory of second order elliptic differential equations discuss the regularity of inequalities of the form

$$\triangle u \ge -f\,u \tag{5}$$

where $f \ge 0$. Encouragingly, we find the following typical result:

Theorem 1.1 *If $\Omega \subset \mathbb{R}^n$ is a pre-compact domain, if $u \in L^2(\Omega)$, if*

$$\triangle u \ge -fu, \quad f \ge 0 \tag{6}$$

and if $f \in L^p$ for some $p > \frac{n}{2}$, then

$$u \in L^\infty \tag{7}$$

where $\|u\|_{L^\infty}$ has a uniform bound depending on n, p, and $\|f\|_{L^p}$.

Referring to our equation (4), if we set $f = u$, analysis holds out the possibility that some L^p bound on $u = C|\,\mathrm{Rm}\,|$ implies *uniform pointwise sectional curvature bounds.*

Now the minimization of a functional might itself provide such a bound, but one notices that the scale-invariant functionals we have considered can only provide bounds at the critical value $p = \frac{n}{2}$, or else on $p = 2$ which is of little help when $n \geq 4$. Indeed the 4-dimensional case is the critical case for this reason. Analysis does indeed give sectional curvature bounds a.e., along with the phenomenon of "concentration of curvature." We shall explore the details and implications in the following lectures.

2 Einstein metrics

Throughout these lectures we shall emphasize the Einstein case. Indeed Einstein metrics are the best understood of the canonical metrics, and the theory is the most complete.

In the 2-dimensional case, the situation is very well understood. Although the equation

$$\mathrm{Ric} \;=\; \frac{R}{2} g \tag{8}$$

is vacant, it is known that if $R = const$ is imposed, then such a metric can always be found on any compact differentiable 2-manifold. After scaling, we may assume $R \in \{+1, 0, -1\}$. The Gauss-Bonnet formula

$$\chi(\Sigma) \;=\; \frac{1}{2\pi} \int R \, dVol \tag{9}$$

imposes volume control on such manifolds (although no diameter control).

The stuation here is very well understood: every closed Riemannian 2-manifold admits a metric of sectional curvature $+1$, 0, or -1, as its Euler number is greater than zero, zero, or less than zero, respectively. As we have seen, the sphere, and therefore the real projective plane, have trivial Riemannian moduli spaces $Mod(M)$. The Torus and Klein bottles each admit 2-parameter moduli spaces. A surface Σ of negative Euler number, however, admits families of metrics of sectional curvature -1 (metrics of constant negative sectional curvature are in 1-1 correspondance with complex structures on such a manifold; to learn more, consult the theory of uniformization of surfaces). Further, the moduli spaces can be non-compact, and "cusp" singularities can form in the limit.

The 3-dimensional case is likewise tractible. Einstein metrics are Ricci-constant, and therefore have constant sectional curvature. In addition, in the negative case, moduli are trivial. The question of which manifolds admit such metrics has been answered elsewhere.

The 4-dimensional case is currently at the center of a great deal of activity. It is well-known that the Riemann tensor splits into four orthogonal pieces:

$$\mathrm{Rm} \;=\; \frac{R}{24} g \circ g \;+\; \frac{1}{n-2} \overset{\circ}{\mathrm{Ric}} \circ g \;+\; |W|^{+} \;+\; |W|^{-} \tag{10}$$

where $\circ$ is meant to denote the Kulkarni-Nomizu product. The tensors W^{+} and W^{-} are orthogonal components of the Weyl tensor. They are characterized as follows. Recall that the Hodge star operator $*$ is an idempotent on $\bigwedge^2$, so that is eigenvalues are ± 1. Therefore

$$\bigwedge^2 \;=\; \bigwedge^{+} \oplus \bigwedge^{-}. \tag{11}$$

Because $*$ is an orthogonal map, we have that $\left\langle \bigwedge^{+}, \bigwedge^{-} \right\rangle = 0$. If Rm and W are regarded as linear maps $\bigwedge^2 \to \bigwedge^2$, we have

$$*W \;=\; W* \tag{12}$$

so that W preserves the $\bigwedge^{+}$ and $\bigwedge^{-}$ eigenspaces. We simply set $W^{+} = W\big|_{\bigwedge^{+}}$ and $W^{-} = W|_{\bigwedge^{-}}$.

The Chern-Gauss-Bonnet formulae in dimension 4 read

$$\begin{aligned} \chi(M) \;&=\; \frac{1}{8\pi^2} \int \frac{1}{24} R^2 \;-\; \frac{1}{2} |\overset{\circ}{\mathrm{Ric}}|^2 \;+\; |W|^2 \\ \tau(M) \;&=\; \frac{1}{12\pi^2} \int |W^{+}|^2 \;-\; |W^{-}|^2 \end{aligned} \tag{13}$$

provide a great deal of information. For instance, the L^2 norm of the Riemann tensor is

$$|\mathrm{Rm}|^2 \;=\; \frac{1}{6} R^2 \;+\; 2 |\overset{\circ}{\mathrm{Ric}}|^2 \;+\; |W|^2. \tag{14}$$

In the case of compact Einstein 4-anifolds, $L^2(|\mathrm{Rm}|)$ is uniformly bounded by $\chi(M)$! In addition,

$$\begin{aligned} 2\chi(M) \;+\; 3\tau(M) \;&=\; \frac{1}{4\pi^2} \int \frac{1}{24} R^2 + 2|W^{+}|^2 \;\geq\; 0 \\ 2\chi(M) \;-\; 3\tau(M) \;&=\; \frac{1}{4\pi^2} \int \frac{1}{24} R^2 + 2|W^{-}|^2 \;\geq\; 0. \end{aligned} \tag{15}$$

The inequality $2\chi \pm 3\tau \geq 0$ is known as the *Hitchin-Thorpe inequality* (in addition, equality occurs only when M is flat, or a quotient of a Calabi-Yau manifold).

3 The Sobolev Inequality: the nexus of geometry and analysis

If Ω is an n-dimensional domain with a Riemannian metric and $\nu > 0$, we define the ν-isoperimetric constant of Ω to be

$$I_\nu(\Omega) \;=\; \inf_{\Omega' \subset\subset \Omega} \frac{\mathrm{Area}(\partial\Omega')}{\mathrm{Vol}(\Omega')^{\frac{\nu-1}{\nu}}}$$

where Area indicates Hausdorff $(n-1)$-measure. If Ω is a closed Riemannian manifold, we take the infimum over domains Ω' with $\mathrm{Vol}\,\Omega' \leq \frac{1}{2}\mathrm{Vol}\,\Omega$; if some such restriction is not made then of course the infimum is zero. Note that if $\nu < n$ then $I_\nu(\Omega) = 0$.

On the other hand we define the ν-Sobolev constant of Ω by

$$S_\nu(\Omega) \;=\; \inf_{f \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla f|}{\left(\int_\Omega |f|^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}}}.$$

If Ω is a closed Riemannian manifold, we take the infimum over functions with $\mathrm{Vol}(\mathrm{supp}\, f) < \frac{1}{2}\mathrm{Vol}(\Omega)$; if some such restriction is not made then of course the infimum is zero.

Theorem 3.1 (Federer-Fleming)

$$I_\nu(\Omega) \;=\; S_\nu(\Omega).$$

Pf

Pf that $S_\nu(\Omega) \leq I_\nu(\Omega)$.
With

$$\int |\nabla f| \;\geq\; S_\nu(\Omega)\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}},$$

we can let $f \equiv 1$ on Ω', $f \equiv 0$ outside $\Omega'^{(\epsilon)}$ (the ϵ-thickening of Ω'), and $f(p) = 1 - \epsilon^{-1}\,\mathrm{dist}(\Omega', p)$ on $\Omega'^{(\epsilon)} - \Omega'$. As $\epsilon \searrow 0$ we have

$$\lim_{\epsilon \searrow 0}\left(\int f^{\frac{\nu}{\nu-1}}\right)^{\frac{\nu-1}{\nu}} \;=\; \mathrm{Vol}(\Omega')^{\frac{\nu-1}{\nu}}$$

$$\lim_{\epsilon \searrow 0}\int |\nabla f| \;=\; \lim_{\epsilon \searrow 0}\frac{\mathrm{Vol}(\Omega'^{(\epsilon)} - \Omega')}{\epsilon} \;=\; \mathrm{Area}(\partial\Omega').$$

Therefore

$$\mathrm{Area}(\partial\Omega') \;=\; \lim_{\epsilon \searrow 0}\int |\nabla f| \;\geq\; \lim_{\epsilon \searrow 0} S_\nu(\Omega)\left(\int f^{\frac{\nu-1}{\nu}}\right)^{\frac{\nu}{\nu-1}} \;=\; S_\nu(\Omega)\,\mathrm{Vol}(\Omega')^{\frac{\nu}{\nu-1}}.$$

Pf that $I_\nu(\Omega) \leq S_\nu(\Omega)$.

Given a nonnegative C_c^∞ function $f : \Omega \to \mathbb{R}$ and given a number t, let $A_t = f^{-1}(t)$ and let $\Omega_t = f^{-1}([t, \infty])$. Locally (near a regular point of f) we can parametrize Ω' by letting f be one coordinate, and putting some coordinates on A_t. We can split the cotangent bundle by letting $df/|df|$ be one covector in an orthonormal coframe. Then if $d\sigma_t$ indicates the wedge product of the remaining vectors, we have Then $dV = \frac{1}{|\nabla f|} df \wedge d\sigma_t$. Therefore

$$\begin{aligned} \int_M |\nabla f| \, dV &= \int_{min(f)}^{max(f)} \int_{A_t} d\sigma_t \, df \;=\; \int_0^\infty \mathrm{Area}(A_t) \, dt \\ &\geq I_\nu(\Omega) \int_0^\infty \mathrm{Vol}(\Omega_t)^{\frac{\nu-1}{\nu}} \, dt \end{aligned}$$

The equality $\int_M |\nabla f| \, dV = \int_0^\infty \mathrm{Area}(A_t) \, dt$ is called the *coarea formula*. Changing the order of integration, *á la* calculus III, gives

$$\begin{aligned} \int f^{\frac{\nu}{\nu-1}} &= \frac{\nu}{\nu-1} \int_\Omega \int_0^{f(p)} t^{\frac{1}{\nu-1}} \, dt \, dVol(p) \\ &= \frac{\nu}{\nu-1} \int_0^\infty \int_{\Omega_t} t^{\frac{1}{\nu-1}} \, dV \, dt \;=\; \frac{\nu}{\nu-1} \int_0^\infty t^{\frac{1}{\nu-1}} \, \mathrm{Vol}(\Omega_t) \, dt \end{aligned}$$

The result follows from the following lemma.

Lemma 3.2 *If $g(t)$ is a nonnegative decreasing function and $s \geq 1$, then*

$$\left(s \int_0^\infty t^{s-1} g(t) \, dt \right)^{\frac{1}{s}} \leq \int_0^\infty g(t)^{\frac{1}{s}} \, dt$$

Pf

We have

$$\begin{aligned} \frac{d}{dT} \left(s \int_0^T t^{s-1} g(t) \, dt \right)^{\frac{1}{s}} &= T^{s-1} g(T) \left(s \int_0^T t^{s-1} g(t) \, dt \right)^{\frac{1}{s}-1} \\ &\leq T^{s-1} g(T)^{\frac{1}{s}} \left(s \int_0^T t^{s-1} \, dt \right)^{\frac{1}{s}-1} \;=\; g(T)^{\frac{1}{s}}. \end{aligned}$$

Since $\frac{d}{dT} \int_0^T g(t)^{\frac{1}{s}} \, dt = g(T)^{\frac{1}{s}}$, we have

$$\left(s \int_0^T t^{s-1} g(t) \, dt \right)^{\frac{1}{s}} \leq \int_0^T g(t)^{\frac{1}{s}} \, dt$$

for all T. $\square$

A deep connection to Ricci curvature can be found in the following theorem due to Chris Croke.

Theorem 3.3 (Croke (1980)) *Assume* (M^n, g) *is a compact Riemannian manifold with Diam* $\leq D$, $\mathrm{Vol}(M) \geq \nu$, *and* $|\mathrm{Ric}| \leq \Lambda$. *There exists a constnat* $C = C(n, D, \nu, \Lambda)$ *so that whenever* $\mathrm{Vol}(\Omega) \leq \frac{1}{2}\mathrm{Vol}(M)$, *we have the following bound on the scale-invariant isoperimetric constant:*

$$I_n(\Omega) \;\leq\; C. \tag{16}$$