Lecture 3 - Epsilon Regularity

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1 Distance functions on $Mod(M)$

One distance function is the Gromov-Hausdorff distance. This will be discussed elsewhere, so we won't dwell, but briefly it is a distnace function on closed metric spaces, defined to be

$$\mathrm{dist}_{GH}(M_1,\, M_2) \;=\; \sup_{x_1 \in M_1, x_2 \in M_2, Z} \mathrm{dist}_Z(x_1, x_2) \tag{1}$$

where the infimum is taken over not just all $x_i \in M_i$, but over all possible embeddings $M_1, M_2 \hookrightarrow Z$ as Z varies over all possible metric spaces that allow such embeddings.

It is best to restrict the Gromov-Hausdorff metric to compact manifolds-with-boundary M. If, for instance, closed but non-compact manifolds are allowed, in fact the topology induced by the Gromov-Hausdorff distance is not only non-compact, but not locally compact, nor even locally paracompact! And that is in addition to the fact that distances between metric spaces might be infinite. If we make the stated restriction however, not only is dist_{GH} a true metric, then although the topology is not compact, it is locally compact.

Generally speaking, if $f : M \to N$ is a map between compact metric spaces (M, d_M) and (N, d_N), we define the *dilation constant* $dil(f)$ of the map to be

$$dil(f) \;=\; \sup_{x \neq y \in M} \frac{d_N(f(x),\, f(y))}{d_M(x,\, y)}. \tag{2}$$

If we are looking at metrics on the same underlying manifold (or manifold-with-boundary), we can define what is called the *Lipschitz distance* on $Met(M)$, by letting d_1, d_2 be the distance functions associated to the two metrics g_1, g_2, and letting f be the identity map from (M, d_1) to $(M,\, d_2)$ as follows

$$\begin{aligned} Lip(g_1,\, g_2) \;&=\; \left|\log \frac{dist_2(x,y)}{dist_1(x,y)}\right| + \left|\log \frac{dist_1(x,y)}{dist_2(x,y)}\right| \\ &=\; \left|\log\, dil(f)\right| + \left|\log\, dil(f^{-1})\right| \end{aligned} \tag{3}$$

This is not diffeomorphism invariant. To fix this, we take the infimum not only over $x, y \in M$ but over diffeomorphisms $\varphi : (M, g_1) \to (M, g_2)$. We have the definition

$$Lip(g_1,\, g_2) \;=\; \sup_{\varphi : M \to M} |\log\, dil(\varphi)| \,+\, \left|\log\, dil(\varphi^{-1})\right| \tag{4}$$

which is a distance function on $Mod(M)$.

We also need the $C^{k,\alpha}$ and $L^{k,p}$ topologies on $Mod(M)$. If g_i is a sequence of Riemannian metrics on M converges is the $C^{k,\alpha}$ topology if, whenever $\varphi : \Omega \to M$ is a differentiable parametrization of some neighborhood im M, the pullback metrics $\varphi^* g_i$ converge in the topology $C^{k,\alpha}(\Omega)$. Notice that this does not provide any distance functions on $Mod(M)$, only a topology.

2 Compactness and Pre-Compactness Theorems

For future reference, we present a pre-compactness result due to Gromov, and a compactness result due to Cheeger.

Theorem 2.1 (Gromov' Precompactness Theorem) *If $\Lambda, D > 0$ and (M_i^n, g_i) is a sequence of Riemannian manifolds with $\mathrm{Ric} \geq -\Lambda$ and $Diam(M) <\leq D$, then there is a metric space (M_∞, d_∞) so that, after passing to a subsequence, we have that*

$$(M_i,\, D_i) \;\to\; (M_\infty,\, d_\infty) \tag{5}$$

in the Gromov-Hausdorff topology. Further, $(M_\infty,\, d_\infty)$ is a length space.

Theorem 2.2 (Cheeger Diffeofiniteness) *Given i_0, D_0, and $\Lambda \geq 0$, then if $\{(M_\alpha^n,\, g_\alpha) \in \alpha \in A$ is the set of all Riemannian manifolds with $\mathrm{inj}_{g_i} M_i \geq i_0$, $Diam_{g_i}(M_i) \leq D$, and $|\,\mathrm{Rm}\,| \leq \Lambda$, there are only finitely many diffeomorphisms classes of manifolds in $\{M_\alpha\}_{\alpha \in A}$*

Theorem 2.3 (eg. Greene-Wu (1988)) *Given i_0, D_0, and $\Lambda \geq 0$, assume $(M_i^n,\, g_i)$ is a sequence of Riemannian manifolds with $\mathrm{inj}_{g_i} M_i \geq i_0$, $Diam_{g_i}(M_i) \leq D$, and $|\,\mathrm{Rm}\,| \leq \Lambda$. Then a smooth differentiable Riemannian manifold $(M_\infty,\, g_\infty)$ exists with g_∞ of class $C^{1,\alpha}$, any $\alpha \in (0, 1)$, so that, after passing to a subsequence, we have that*

$$(M_i,\, g_i) \;\to\; (M_\infty,\, g_\infty) \tag{6}$$

in the Gromov-Hausdorff, Lipschitz topologies, and $C^{1,\alpha}$ topologies.

Lastly, it is possible to relax the condition on injectivity radii:

Theorem 2.4 (Cheeger's Lemma) *Given positive n, ν, D, Λ, there exists a constant $i_0 = i_0(n, \nu,\, D,\, \Lambda)$ so that if (M^n, g) is a Riemannian manifold with $\mathrm{Vol}\ M \geq \nu$, $Diam\, M \leq D$, and $|\,\mathrm{Rm}\,| \leq \Lambda$, then $\mathrm{inj}\ M \geq i_0$.*

3 Epsilon-Regularity

Lemma 3.1 *If $f \geq 0$ is any weakly differentiable function on a Riemannian manifold (M, g) and φ is any $C^{0,1}$ function with compact support, then*

$$\int \varphi^2 u^{p-2} |\nabla u|^2 \;\leq\; \left(\frac{2}{p-1}\right)^2 \int |\nabla \varphi|^2 u^p \;-\; \frac{2}{p-1} \int \varphi^2 u^{p-1} \triangle u. \tag{7}$$

for any $p > 1$.

Proof. We have

$$\begin{aligned}(p-1)\int \varphi^2 u^{p-2}|\nabla u|^2 \;&=\; \int \varphi^2 \left\langle \nabla u^{p-1},\, \nabla u \right\rangle \\ &=\; \int div\left(\varphi^2 u^{p-1} \nabla u\right) \;-\; 2\int \varphi^2 u^{p-1} \left\langle \varphi,\, \nabla u \right\rangle \;-\; \int \varphi^2 u^{p-1} \triangle u \\ &=\; \int \left\langle \frac{-2}{\sqrt{p-1}} u^{p/2} \varphi,\; \sqrt{p-1} u^{\frac{p-2}{2}} \nabla u \right\rangle \;-\; \int \varphi^2 u^{p-1} \triangle u\end{aligned} \tag{8}$$

Using the Cauchy-Schwarz inequality on the last expression we get

$$\begin{aligned}&(p-1)\int \varphi^2 u^{p-2}|\nabla u|^2 \\ &\leq\; \frac{2}{p-1}\int |\nabla\varphi|^2 u^p \;+\; \frac{p-1}{2}\int \varphi^2 u^{p-2}|\nabla u|^2 \;-\; \int \varphi^2 u^{p-1} \triangle u\end{aligned} \tag{9}$$

so that

$$\frac{p-1}{2}\int \varphi^2 u^{p-2}|\nabla u|^2 \;\leq\; \frac{2}{p-1}\int |\nabla\varphi|^2 u^p \;-\; \int \varphi^2 u^{p-1}\triangle u \tag{10}$$

as required. □

Corollary 3.2 *If (M, g) is a Riemannian manifold with an Einstein metric and $u = C|\mathrm{Rm}|$ (where $C = C(n)$ was chosen previously), then u is Lipschitz and*

$$\int \varphi^2 u^{p-2}|\nabla u|^2 \;\leq\; \left(\frac{2}{p-1}\right)^2 \int |\nabla\varphi|^2 u^p \;+\; \frac{2}{p-1}\int \varphi^2 u^{p+1}. \tag{11}$$

whenever φ is a $C^{0,1}$ function with compact support.

It is convenient to set

$$\gamma \;=\; \frac{n}{n-2}. \tag{12}$$

Lemma 3.3 (Main Lemma) *For any $p > 1$ we have*

$$\left(\int \varphi^{2\gamma} u^{p\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; 2S_n^2 \left(1 + \frac{p^2}{(p-1)^2}\right) \int |\nabla\varphi|^2 u^p \;+\; S_n^2 \frac{p^2}{p-1} \int \varphi^2 u^{p+1} \tag{13}$$

Proof. The Sobolev inequality gives

$$\begin{aligned}
\left(\int f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &= \left(\int |f^{\frac{2(n-1)}{n-2}}|^{\frac{n}{n-1}}\right)^{\frac{n-2}{n}} \\
&\leq \left(S_n \int |\nabla f^{\frac{2(n-1)}{n-2}}|\right)^{\frac{n-2}{n-1}} \\
&\leq \left(S_n \left(\frac{2(n-1)}{n-2}\right) \int f^{\frac{2(n-1)-n+2}{n-2}} |\nabla f|\right)^{\frac{n-2}{n-1}} \\
&= \left(S_n \left(\frac{2(n-1)}{n-2}\right) \int f^{\frac{n}{n-2}} |\nabla f|\right)^{\frac{n-2}{n-1}} \\
&= \left(S_n \left(\frac{2(n-1)}{n-2}\right) \left(\int f^{\frac{2n}{n-2}}\right)^{\frac{1}{2}} \left(\int |\nabla f|^2\right)^{\frac{1}{2}}\right)^{\frac{n-2}{n-1}}
\end{aligned} \tag{14}$$

so that

$$\begin{aligned}
\left(\int f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n} - \frac{1}{2}\frac{n-2}{n-1}} &\leq \left(S_n \left(\frac{2(n-1)}{n-2}\right)\right)^{\frac{n-2}{n-1}} \left(\int |\nabla f|^2\right)^{\frac{1}{2}\frac{n-2}{n-1}} \\
\left(\int f^{\frac{2n}{n-2}}\right)^{\frac{(n-2)^2}{2n(n-1)}} &\leq \left(S_n \left(\frac{2(n-1)}{n-2}\right)\right)^{\frac{n-2}{n-1}} \left(\int |\nabla f|^2\right)^{\frac{1}{2}\frac{n-2}{n-1}} \\
\left(\int f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} &\leq S_n \left(\frac{2(n-1)}{n-2}\right) \left(\int |\nabla f|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{15}$$

From the Sobolev inequality we have

$$\begin{aligned}
\left(\int \varphi^{2\gamma} u^{p\gamma}\right)^{\frac{1}{\gamma}} &\leq S_n^2 \int |\nabla(\varphi u^{\frac{p}{2}})|^2 \\
&\leq 2S_n^2 \int |\nabla\varphi|^2 u^p \;+\; S_n^2 \frac{p^2}{2} \int \varphi^2 u^{p-2} |\nabla u|^2 \\
&\leq 2S_n^2 \left(1 + \frac{p^2}{(p-1)^2}\right) \int |\nabla\varphi|^2 u^p \;+\; S_n^2 \frac{p^2}{p-1} \int \varphi^2 u^{p+1}
\end{aligned} \tag{16}$$

□

Lemma 3.4 *If*

$$\int_{supp\varphi} u^{\frac{n}{2}} \;\leq\; \frac{1}{2}\left(S_n^2 \frac{n}{2} \frac{n}{n-2}\right)^{-1} \tag{17}$$

then

$$\left(\int \varphi^{2\gamma} u^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; C\, S_n^2 \int |\nabla\varphi|^2 u^{\frac{n}{2}} \tag{18}$$

where $C = \frac{8n^2-16n+16}{(n-2)^2}$.

Pf. For any $p > 1$ we obtain

$$\begin{aligned} \left(\int \varphi^{2\gamma} u^{p\gamma}\right)^{\frac{1}{\gamma}} \;&\leq\; S_n^2 \int |\nabla(\varphi u^{\frac{p}{2}})|^2 \\ &\leq\; 2S_n^2 \int |\nabla\varphi|^2 u^p \;+\; S_n^2 \frac{p^2}{2} \int \varphi^2 u^{p-2} |\nabla u|^2 \\ &\leq\; 2S_n^2 \left(1 + \frac{p^2}{(p-1)^2}\right) \int |\nabla\varphi|^2 u^p \;+\; S_n^2 \frac{p^2}{p-1} \int \varphi^2 u^{p+1} \end{aligned} \tag{19}$$

We can use Hölder's inequality on the last piece:

$$\int \varphi^2 u^{p+1} \;\leq\; \left(\int \varphi^{2\gamma} u^{p\gamma}\right)^{\frac{1}{\gamma}} \left(\int_{supp\varphi} u^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{20}$$

Now setting $p = \frac{n}{2}$ we get

$$\begin{aligned} \left(\int \varphi^{2\gamma} u^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; 2S_n^2 \left(1 + \frac{n^2}{(n-2)^2}\right) \int |\nabla\varphi|^2 u^{\frac{n}{2}} \\ +\; S_n^2 \frac{n}{2} \frac{n}{n-2} \left(\int_{\operatorname{supp}\varphi} u^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int \varphi^{2\gamma} u^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \end{aligned} \tag{21}$$

Therefore

$$\int_{supp\varphi} u^{\frac{n}{2}} \;\leq\; \frac{1}{2} \left(S_n^2 \frac{n}{2} \frac{n}{n-2}\right)^{-1} \tag{22}$$

implies

$$\left(\int \varphi^{2\gamma} u^{\frac{n}{2}\gamma}\right)^{\frac{1}{\gamma}} \;\leq\; 4S_n^2 \left(1 + \frac{n^2}{(n-2)^2}\right) \int |\nabla\varphi|^2 u^{\frac{n}{2}} \tag{23}$$

□

If we are even smarter in our application of Hölder's inequality, we can do better yet. Notice that

$$1 \;=\; \frac{1}{\gamma^2} \;+\; \frac{2}{n}\frac{1}{\gamma} \;+\; \frac{2}{n}, \tag{24}$$

so setting

$$\varphi^2 u^{p+1} \;=\; \varphi^{2\cdot\frac{1}{\gamma}} u^{p\frac{1}{\gamma}} \cdot \varphi^{2\cdot\frac{2}{n}} u \cdot u^{p\frac{2}{n}} \tag{25}$$

we obtain

$$\begin{aligned} p\int \varphi^2 u^{p+1} \;&\le\; p\left(\int \varphi^{2\gamma}u^{p\gamma}\right)^{\frac{1}{\gamma^2}} \left(\int_{supp\varphi} u^{\frac{n}{2}\gamma}\right)^{\frac{2}{n}\frac{1}{\gamma}} \left(\int \varphi^2 u^p\right)^{\frac{2}{n}} \\ &\le\; p\left(\int \varphi^{2\gamma}u^{p\gamma}\right)^{\frac{1}{\gamma^2}} \left(C\,S_n^2 \sup|\nabla\varphi|^2 \int_{supp\varphi} u^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int \varphi^2 u^p\right)^{\frac{2}{n}} \\ &\le\; p\left(\int \varphi^{2\gamma}u^{p\gamma}\right)^{\frac{1}{\gamma^2}} \left(C \sup|\nabla\varphi|^2\right)^{\frac{2}{n}} \left(\int \varphi^2 u^p\right)^{\frac{2}{n}} \\ &\le\; \frac{n-2}{n}\left(\int \varphi^{2\gamma}u^{p\gamma}\right)^{\frac{1}{\gamma}} \;+\; C\,p^{\frac{n}{2}} \sup|\nabla\varphi|^2 \int \varphi^2 u^p \end{aligned} \tag{26}$$

for a different $C = C(n)$.

Plugging into above we have

$$\left(\int \varphi^{2\gamma}u^{p\gamma}\right)^{\frac{1}{p\gamma}} \;\le\; C^{\frac{1}{p}}\;\left(1+p^{\frac{n}{2}}\right)^{\frac{1}{p}} S_n^{\frac{2}{p}} \left(\sup|\nabla\varphi|\right)^{\frac{2}{p}} \left(\int_{\mathrm{supp}\,\varphi} u^p\right)^{\frac{1}{p}}. \tag{27}$$

Consider the ball $B_q(r)$, and let S_n be the Sobolev constant on $B_q(r)$. Set $r_i = \frac{r}{2}\left(1+2^{-i}\right)$, and choose a collection of functions φ_i so that $\varphi_i = 1$ on $B_q(r_{i+1})$, $\varphi_i = 0$ on $M\setminus B_q(r_i)$, and so $|\nabla\varphi_i| \le \frac{r}{8}2^{-i}$. Replacing p with $p\gamma^i$ we get

$$\begin{aligned} \left(\int \varphi_i^{2\gamma^{i+1}} u^{p\gamma^{i+1}}\right)^{\frac{1}{p\gamma^{i+1}}} \;&\le\; C^{\frac{1}{p\gamma^i}}\;\left(p^{\frac{n}{2}}\gamma^{i\frac{n}{2}}\right)^{\frac{1}{p\gamma^i}} S_n^{\frac{2}{\gamma^i}} \left(\sup|\nabla\varphi|\right)^{\frac{2}{p\gamma^i}} \left(\int_{\mathrm{supp}\,\varphi} u^{p\gamma^i}\right)^{\frac{1}{p\gamma^i}} \\ \left(\int_{B_q(r_{i+1}))} u^{p\gamma^{i+1}}\right)^{\frac{1}{p\gamma^{i+1}}} \;&\le\; C^{\frac{1}{p\gamma^i}}\;\left(p^{\frac{n}{2}}\gamma^{i\frac{n}{2}}\right)^{\frac{1}{p\gamma^i}} S_n^{\frac{2}{p\gamma^i}} \left(r^{-1}\,2^i\right)^{\frac{2}{p\gamma^i}} \left(\int_{B_q(r_i)} u^{p\gamma^i}\right)^{\frac{1}{p\gamma^i}}. \end{aligned} \tag{28}$$

Set

$$\Phi_i \;=\; \left(\int_{B_q(r_i)} u^{p\gamma^i}\right)^{\frac{1}{p\gamma^i}} \tag{29}$$

Note that $\Phi_\infty = \lim_i \Phi_i \;=\; sup_{B_q(r/2)}|u|$, and $\Phi_0 = \left(\int_{B_q(r)} u^p\right)^{\frac{1}{p}}$.

We obtain

$$\begin{aligned} \Phi_{N+1} \;&\le\; \left(\prod_{i=0}^{N}\left(C_n^{\frac{1}{p}} S_n^{\frac{2}{p}} r^{\frac{2}{p}}\right)^{\gamma^{-i}} \left(2^{\frac{2}{p}}\right)^{i\gamma^{-i}}\right)\Phi_0 \\ &=\; \left(C_n^{\frac{1}{p}} S_n^{\frac{2}{p}} r^{-\frac{2}{p}}\right)^{\sum_{i=0}^{N}\gamma^{-i}} \left(2^{\frac{2}{p}}\right)^{\sum_{i=0}^{N} i\gamma^{-i}}\;\Phi_0 \end{aligned} \tag{30}$$

4 Exercises

1) Prove that Lip on $Mod(M)$ is a distance function.

2) Letting g_i be a sequence of metrics on the compact manifold M^n, show that the g_i converge in the topology induced by the Lipschitz distance if they converge in the C^0 topology.