Lecture 4 - Convergence

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1 Hypotheses

How can sequences of manifolds with a canonical metric degenerate? In the Einstein case, under conditions on energy, diameter, and volume, we will see that they degenerate, at worst, to manifolds with point-like singularities, which are of orbifold type.

Higher dimensional case: $\int |\operatorname{Rm}|^{\frac{n}{2}} < \Lambda$, $\operatorname{Diam}(M) < D$, $\operatorname{Vol}(M) \geq \nu$, $\operatorname{Ric} \in \{-1, 0, 1\}$.

4-dimensional Einstein case: $\chi(M) < \Lambda$, $\operatorname{Diam}(M) < D$, $\operatorname{Vol}(M) \geq \nu$, $\operatorname{Ric} \in \{-1, 0, 1\}$. If $\operatorname{Ric} = +1$ we can eliminate the diamater bound.

2 Distance functions on $Mod(M)$

One distance function is the Gromov-Hausdorff distance. This will be discussed elsewhere, so we won't dwell, but briefly it is a distnace function on closed metric spaces, defined to be

$$\operatorname{dist}_{GH}(M_1, M_2) \;=\; \inf_Z \left(\sup_{x_1 \in M_1} \inf_{x_2 \in M_2} \operatorname{dist}_Z(x_1, x_2) + \sup_{x_2 \in M_2} \inf_{x_1 \in M_1} \operatorname{dist}_Z(x_1, x_2) \right) \tag{1}$$

where the infimum is taken over not just all $x_i \in M_i$, but over all possible embeddings $M_1, M_2 \hookrightarrow Z$ as Z varies over all possible metric spaces that allow such embeddings.

It is best to restrict the Gromov-Hausdorff metric to compact manifolds-with-boundary M. If, for instance, closed but non-compact manifolds are allowed, in fact the topology induced by the Gromov-Hausdorff distance is not only non-compact, but not locally compact, nor even locally paracompact! And that is in addition to the fact that distances between metric spaces might be infinite. If we make the stated restriction however, not only is dist_{GH} a true metric, then although the topology is not compact, it is locally compact.

Generally speaking, if $f : M \to N$ is a map between compact metric spaces (M, d_M)

and (N, d_N), we define the *dilation constant* $dil(f)$ of the map to be

$$dil(f) \;=\; \sup_{x\neq y\in M} \frac{d_N(f(x),\, f(y))}{d_M(x,\, y)}. \tag{2}$$

If we are looking at metrics on the same underlying manifold (or manifold-with-boundary), we can define what is called the *Lipschitz distance* on $Met(M)$, by letting $d_1,\ d_2$ be the distance functions associated to the two metrics $g_1,\ g_2$, and letting Id be the identity map from (M, d_1) to $(M,\ d_2)$ as follows

$$\begin{aligned} Lip(g_1,\, g_2) \;&=\; \sup_{x\neq y}\left|\log\frac{dist_2(x,y)}{dist_1(x,y)}\right| \,+\, \sup_{x\neq y}\left|\log\frac{dist_1(x,y)}{dist_2(x,y)}\right| \\ &=\; \left|\log\, dil(Id)\right| \,+\, \left|\log\, dil(Id^{-1})\right| \end{aligned} \tag{3}$$

This is not diffeomorphism invariant. To fix this, we take the infimum not only over $x, y \in M$ but over diffeomorphisms $\varphi : (M, g_1) \to (M, g_2)$. We have the definition

$$Lip(g_1,\, g_2) \;=\; \sup_{\varphi:M\to M} \left|\log\, dil(\varphi)\right| \,+\, \left|\log\, dil(\varphi^{-1})\right| \tag{4}$$

which is a distance function on $Mod(M)$.

We also need the $C^{k,\alpha}$ and $L^{k,p}$ topologies on $Mod(M)$. If g_i is a sequence of Riemannian metrics on M converges is the $C^{k,\alpha}$ topology if, whenever $\varphi : \Omega \to M$ is a differentiable parametrization of some neighborhood in M, the pullback metrics $\varphi^* g_i$ converge in the topology $C^{k,\alpha}(\Omega)$. Notice that this does not provide any distance functions on $Mod(M)$, only a topology.

3 Compactness and Pre-Compactness Theorems

For future reference, we present a pre-compactness result due to Gromov, and a compactness result due to Cheeger.

Theorem 3.1 (Gromov' Precompactness Theorem) *If $\Lambda, D > 0$ and (M_i^n, g_i) is a sequence of Riemannian manifolds with $\mathrm{Ric} \geq -\Lambda$ and $Diam(M) \leq D$, then there is a metric space (M_∞, d_∞) so that, after passing to a subsequence, we have that*

$$(M_i,\, d_i) \;\to\; (M_\infty,\, d_\infty) \tag{5}$$

in the Gromov-Hausdorff topology. Further, $(M_\infty,\, d_\infty)$ is a length space.

Theorem 3.2 (Cheeger Diffeofiniteness) *Given i_0, D_0, and $\Lambda \geq 0$, then if $\{(M_\alpha^n,\, g_\alpha)\}_{\alpha\in A}$ is the set of all Riemannian manifolds with $\mathrm{inj}_{g_i} M_i \geq i_0$, $Diam_{g_i}(M_i) \leq D$, and $|\,\mathrm{Rm}\,| \leq \Lambda$, then the manifolds fall into one of only finitely many diffeomorphisms types.*

Theorem 3.3 (eg. Greene-Wu (1988)) *Given i_0, D_0, and $\Lambda \geq 0$, assume (M_i^n, g_i) is a sequence of Riemannian manifolds with $\mathrm{inj}_{g_i} M_i \geq i_0$, $Diam_{g_i}(M_i) \leq D$, and $|\mathrm{Rm}| \leq \Lambda$. Then a smooth differentiable Riemannian manifold (M_∞, g_∞) exists with g_∞ of class $C^{1,\alpha}$, any $\alpha \in (0,1)$, so that, after passing to a subsequence, we have that*

$$(M_i, g_i) \rightarrow (M_\infty, g_\infty) \tag{6}$$

in the Gromov-Hausdorff, Lipschitz, and $C^{1,\alpha}$ topologies.

Lastly, it is possible to relax the condition on injectivity radii:

Theorem 3.4 (Cheeger's Lemma) *Given positive n, ν, D, Λ, there exists a constant $i_0 = i_0(n, \nu, D, \Lambda)$ so that if (M^n, g) is a Riemannian manifold with $\mathrm{Vol}\, M \geq \nu$, $Diam\, M \leq D$, and $|\mathrm{Rm}| \leq \Lambda$, then $\mathrm{inj}\ M \geq i_0$.*

4 Epsilon-Regularity

Recall the theorem from last time:

Theorem 4.1 (Epsilon-regularity) *There exist constants $C = C(n, S_n)$, $\epsilon_0 = \epsilon_0(n, S_n)$ so that*

$$\left(\int_{B_q(r)} |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \epsilon_0 \tag{7}$$

implies

$$\sup_{B_q(r/2)} |\mathrm{Rm}| \leq C r^{-2} \left(\int_{B_q(r)} |\mathrm{Rm}|^{\frac{n}{2}} \right)^{\frac{2}{n}}. \tag{8}$$

5 Weak Convergence

5.1 The "good" and "bad" sets

Let

$$\mathcal{G}_r = \left\{ p \in M \;\Big|\; \int_{B_p(r)} |\mathrm{Rm}|^{\frac{n}{2}} < \epsilon_0^{\frac{2}{n}} \right\}. \tag{9}$$

$$\mathcal{B}_r \;=\; \left\{ p \in M \;\Big|\; \int_{B_p(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\geq\; \epsilon_0^{\frac{2}{n}} \right\}. \tag{10}$$

be its compliment. We show that $\mathcal{B}_r$ is a set of *small content*, when Ricci curvature has an a priori bound.

Cover $\mathcal{B}_r$ with balls $B_{q_i}(2r)$ of radius $2r$ so that the balls $B_{q_i}(r)$ are disjoint. By definition

$$\left(\int_{B_{q_i}(r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}} \;\geq\; \epsilon_0. \tag{11}$$

Therefore

Lemma 5.1 *If $(M^n,\, g)$ is a Riemannian manifold with*

$$\left(\int_{M} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}} \;\leq\; \Lambda, \tag{12}$$

then $\mathcal{B}_r$ can be covered with at most

$$\Lambda\,\epsilon_0^{-1} \tag{13}$$

many balls of radius $2r$.

Corollary 5.2 *If $(M^n,\, g)$ is a Riemannian manifold with*

$$\left(\int_{M} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}} \;\leq\; \Lambda \tag{14}$$

and $\mathrm{Ric} \geq -1$ *and $r \leq 1$, then*

$$\mathrm{Vol}\;\mathcal{B}_r \;\leq\; C_n \Lambda \epsilon_0^{-1}\, r^n. \tag{15}$$

□

Now proviided $\mathrm{Ric} \geq -1$ we have

$$M \;=\; \mathcal{G}_r \,\cup\, \mathcal{B}_r \tag{16}$$

where

$$|\,\mathrm{Rm}\,| \;\leq\; r^{-2} \quad \textit{on} \quad \mathcal{G}_r \tag{17}$$

and

$$\mathrm{Vol}\;\mathcal{B}_r \;\leq\; C_n \Lambda \epsilon_0^{-1}\, r^n. \tag{18}$$

5.2 Convergence

Consider a sequence of metrics g_i on the Einstein manifolds M_i^4. Let $M_i = \mathcal{B}_{r,i} \cup \mathcal{G}_{r,i}$ be the decomposition from above.

Given r, Cheeger diffeofiniteness gives us finitely many diffeomorphism classes among the

$$\{\mathcal{G}_{r,i}\}_i. \tag{19}$$

There we can consider the differentiable manifolds $\mathcal{G}_{r,i}$ fixed and evolve the metrics. With sectional curvature bounded, we obtian $C^{1,\alpha}$ of the metrics (after passing to a subsequence). The elliptic equality $\triangle \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}$ implies higher regularity convergence: $C^{k,\alpha}$.

Now shrink r a little, and pass to a further subsequence. Letting r_j be a sequence of radii $r_j \searrow 0$, we have sequences

$$\mathcal{G}_{r_j,\, i_j} \tag{20}$$

A diagonal subsequence converges to a Riemannian manifold with at most $\Lambda\epsilon_0^{-1}$ many points removed. Volume is continuous in the limit, and $\int |\,\mathrm{Rm}\,|^2$ is lower semi-continuous.

6 Singularities

Here is a quick overview of what happens in the point-like singularities.

Sectional curvature is increasing unboundedly, but one can scale the metric so that $|\,\mathrm{Rm}\,| \leq 1$, and take a pointed limit.

These limits are Ricci-flat manifolds, with positive asymptotic volume ratio and $\epsilon_0 < \int |\,\mathrm{Rm}\,|^{\frac{n}{2}} < \Lambda$. Clearly $|\,\mathrm{Rm}\,| = o(r^{-2})$, but in fact one can show that $|\,\mathrm{Rm}\,| = O(r^{-2-\delta})$ for some δ. This is enough to rule out critical points of the distance function, so outside some compact subset Ω, topology is trivial: $M \setminus \Omega$ is diffeomorphic to a quotient of a Euclidean annulus. Such a manifold is called ALE; almost locally Euclidean.

Assuming original manifolds were simply connected, the bubbles have trivial b_1. By Poincare duality, $b_3 = 0$; therefore $\chi(bubble) \geq 1$. In fact $\chi(bubble) > 1$—thus these bubbles absorb homology!

$$\chi(Bubble) \;=\; \frac{1}{|\Gamma|} \;+\; \frac{1}{8\pi^2}\int |W|^2 \tag{21}$$