Chapter 12

Imaging artifacts in x-ray tomography

In the previous chapter we derived finite algorithms to approximately reconstruct a function of two variables, with bounded support from finitely many samples of its Radon transform. For a variety of reasons this is still a highly idealized situation. In this chapter we analyze these algorithms with a more realistic model for the measurement process. The first issue we address is the fact that the x-ray beam has a finite (two-dimensional) width. There is a simple linear model for this effect as weighted averaging in the affine parameter. More careful investigation reveals that this is a non-linear phenomenon. Averaging in the affine parameter is a form of low pass filtering which is in turn important for the analysis of the aliasing that results from sampling the Radon transform. We next derive the total “point spread function” for the measurement-reconstruction process first without, and then with sampling. Using the point spread function we consider the consequences for the reconstructed image of various sorts of measurement errors. At the end of the chapter we consider beam hardening; this results from the fact that the x-ray beam is not monochromatic. Beam hardening is a fundamentally non-linear phenomenon.

12.1 The effect of a finite width x-ray beam

Up to now, we have assumed that an x-ray “beam” is just a line with no width and that the measurements are integrals over such lines. What is really measured is better approximated by averages of such integrals. We now consider how the finite width of the x-ray beam affects the measured data. Our treatment closely follows the discussion in [71].

12.1.1 A linear model for finite beam width

A simple linear model for this effect is to replace the line integrals of $f$

$$Rf(t,\omega) = \int_{-\infty}^{\infty} f(t\omega + s\omega)ds,$$

by a weighted average of these line integrals

$$RWf(t,\omega) = \int_{-\infty}^{\infty} w(u)Rf(t - u,\omega)du.$$
Such an average is called a \textit{strip integral}. Here \( w \) is a non-negative weight function; it models both the distribution of energy across the x-ray beam and the detector used to make the measurements, see figure 12.1. This function is sometimes called the \textit{beam profile}, though of course the actual beam profile must incorporate the third dimension as well. The total integral of \( w \) is normalized to be one.

![Figure 12.1: The finite size of the x-ray source and detector means real measurements are modeled by strip integrals.](image)

The relationship between \( R f \) and \( R_W f \) is a consequence of the convolution theorem for the Radon transform. In the imaging literature it is due to Shepp and Logan.

\textbf{Theorem 12.1.1 (Shepp and Logan).} The weighted Radon transform \( R_W f \) is the Radon transform of the convolution, \( f \ast k \) where \( k \) is the radial function

\[
k(x, y) = -\frac{1}{\pi \rho} \partial_{\rho} \left[ \int_{\rho}^{\infty} \frac{w(u)u}{\sqrt{u^2 - \rho^2}} du \right] \bigg|_{\rho = \sqrt{x^2 + y^2}}.
\]

\textit{Remark 12.1.1.} If \( w \) has bounded support then the integrand of \( k \) is zero for sufficiently large \( \rho \), hence \( k(\sqrt{x^2 + y^2}) \) also has bounded support. Similarly if \( k \) has bounded support then so does \( w \).

\textit{Proof.} The theorem is an immediate consequence of Proposition 6.1.1. The function \( R_W f \) is the convolution in the \( t \)-parameter of \( R f \) with \( w \). If \( k \) is a function on \( \mathbb{R}^2 \) such that \( Rk = w \) then the proposition states that

\[
R_W f = R(f \ast k).
\]

Since \( w \) is independent of \( \omega \) it follows that \( k \) must also be a radial function. The formula for \( k \) is the Radon inversion formula for radial functions derived in section 3.5.
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Example 12.1.1. Some simple examples of \((w, k)\)-pairs are

\[
    w^1(u) = \begin{cases} \frac{1}{\sqrt{2\pi}} & u \in [-\delta, \delta], \\ 0 & |u| > \delta \end{cases}, \quad k^1(\rho) = \begin{cases} \frac{1}{\sqrt{2\pi\delta}} \frac{1}{\sqrt{\rho^2 - \rho^2}} & 0 \leq \rho < \delta, \\ 0 & \rho \geq \delta, \end{cases}
\]

and

\[
    w^2(u) = \begin{cases} \frac{1}{\sqrt{\pi\delta^2}} (\delta^2 - u^2)^{1/2} & |u| < \delta, \\ 0 & |u| > \delta \end{cases}, \quad k^2(\rho) = \begin{cases} \frac{1}{\sqrt{\pi\delta^2}} & \rho < \delta, \\ 0 & 0 \leq \rho \geq \delta. \end{cases}
\]

A consequence of finite strip width is that the actual measurements are samples of the Radon transform of \(f \ast k\), which is a somewhat smoothed version of \(f\). Indeed

\[
\widehat{R_W f}(r, \omega) = \hat{w}(r) \widehat{R f}(r, \omega)
\]

and therefore the finite strip width leads to low pass filtering of \(R f\) in the affine parameter. This has the desirable effect of reducing the aliasing artifacts that result from sampling. In x-ray tomography this is essentially the only way to low pass filter the data before it is sampled. Of course this averaging process also leads to a loss of resolution; so the properties of the averaging function \(w\) need to be matched with the sample spacing. As we saw in section 9.1.9 the effects of such averaging, can to some extent be removed, nonetheless algorithms are often evaluated in terms of their ability to reconstruct samples of \(f \ast k\) rather than \(f\) itself.

As mentioned above the beam profile, \(w\) models the effects of the source and detector together. This function is built out of two pieces: the detector response function, \(w_d\) and the source function, \(w_s\). If \(I(u)\) describes the intensity of the (two-dimensional) x-ray beam incident on the (one-dimensional) detector at the point \(u\) then the output of the detector is modeled as

\[
\int_{-\infty}^{\infty} w_d(u) I(u) du.
\]

Suppose that the function \(w_s\) models the source so that the energy of the x-ray beam in the interval \([a, b]\) is given by

\[
\int_{a}^{b} w_s(u) du.
\]

If the source and detector are fixed in space, relative to one another, then the combined effect of this source-detector pair is modeled by the pointwise product \(w = w_s \cdot w_d\). This is the geometry in a third generation, fan beam scanner. In some parallel beam scanners and fourth generation scanners the detectors are fixed and the source moves. In this case a model for the source-detector pair is the convolution \(w = w_s \ast w_d\). The detector is often modeled by a simple function like \(w^1\) or \(w^2\) defined in example 12.1.1, while the source is often described by a Gaussian, \(ce^{-\frac{u^2}{\sigma^2}}\). In this case the x-ray source is said to have a Gaussian focal spot.

Remark 12.1.2. A very thorough treatment of the problem of modeling x-ray sources and detectors is given in [4].

Exercises
Exercise 12.1.1. Explain why these two models of source-detector pairs are reasonable for the different hardware. Assume that the x-rays are non-diverging. In particular, explain how relative motion of the source and detector leads to a convolution.

Exercise 12.1.2. A real x-ray beam is three dimensional. Suppose that the third dimension is modeled as in (11.3) by the slice selectivity profile $w_{ssp}$. Give a linear model for what is measured, analogous to the Shepp-Logan result, which includes the third dimension.

Exercise 12.1.3. What is the physical significance of the total integral of $w$?

12.1.2 A non-linear model for finite beam width

Unfortunately the effect of finite beam width is a bit more complicated than described in the previous section. If we could produce a 1-dimensional x-ray beam then, what we would measure would actually be

$$I_o = I_i \exp[-Rf(t, \omega)],$$

where $I_i$ is the intensity of the x-ray source and $I_o$ is the measured output. For a strip what is actually measured is closer to

$$\frac{I_o}{I_i} \approx \int_{-\infty}^{\infty} w(u) \exp[-Rf(t-u, \omega)] du.$$

Thus the measurement depends non-linearly on the attenuation coefficient. If $w$ is very concentrated near $u = 0$ and $\int w(u) du = 1$ then

$$\log \frac{I_o}{I_i} \approx \int_{-\infty}^{\infty} w(u) Rf(t-u, \omega) du.$$

To derive this expression we use the Taylor expansions:

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O(x^4),$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4).$$

This analysis assumes that the oscillation of $Rf(t-u, \omega)$ over the support of $w$ is small. We begin by factoring out $\exp(-Rf)$:

$$\int_{-\infty}^{\infty} w(u) \exp(-Rf(t-u, \omega)) du = \exp(-Rf(t, \omega)) \int_{-\infty}^{\infty} w(u) \exp[Rf(t, \omega) - Rf(t-u, \omega)] du$$
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Using the Taylor expansion for $e^{-x}$ gives:

$$\int_{-\infty}^{\infty} w(u) \exp(-Rf(t-u,\omega)) du$$

$$= \exp(-Rf(t,\omega)) \int_{-\infty}^{\infty} w(u)[1 - (Rf(t,\omega) - Rf(t-u,\omega)) + O((Rf(t-u,\omega) - Rf(t,\omega))^2)] du$$

$$= \exp(-Rf(t,\omega)) \left[ 1 - \int_{-\infty}^{\infty} w(u)(Rf(t-u,\omega) - Rf(t,\omega)) du + O\left( \int w(u)[Rf(t-u,\omega) - Rf(t,\omega)]^2 du \right) \right]$$

Taking $-\log$, using the Taylor expansion for $\log(1+x)$ and the assumption that $\int w(u) = 1$ gives

$$\log \frac{I_0}{I} \approx \int w(u) Rf(t-u,\omega) du + O\left( \int w(u)[Rf(t-u,\omega) - Rf(t,\omega)]^2 du \right). \quad (12.3)$$

The leading order error is proportional to the mean square oscillation of $Rf$ weighted by $w$.

12.1.3 The partial volume effect

If the variation of $Rf(t,\omega)$ is large over the width of the strip then the error term dominates in (12.3). In practice this happens if part of the x-ray beam intercepts bone and the remainder passes through soft tissue. In imaging applications this is called the partial volume effect. To illustrate this we consider a simple special case. Suppose that the intensity of the x-ray beam is constant across a strip of width 1. Half the strip is blocked by a rectangular object of height 1 with attenuation coefficient 2 and half the strip is empty. If we assume that $w(u) = \chi_{[0,1]}(u)$ then

$$-\log \left[ \int_{-\infty}^{\infty} w(u) \exp[-Rf(t-u,\omega)] du \right] = -\log \left[ \frac{1 + e^{-2}}{2} \right] \approx 0.5662,$$

whereas

$$\int_{-\infty}^{\infty} w(u) Rf(t-u,\omega) du = 1.$$

In the table 12.1 we give the linear and non-linear computations for an absorbent unit square with two attenuation coefficients $\mu_0, \mu_1$ each occupying half, see figure 12.2.
An even more realistic example is provided by a long rectangle of absorbing material with a small inclusion of more absorbent material, as shown in figure (12.3). The graphs in figure 12.4 show the relative errors with \( \mu_0 = 1 \) and \( \mu_1 \in \{1.5, 2, 2.5, 3\} \). This is a model for a long stretch of soft tissue terminating at a piece of bone.

The artifact caused by the partial volume effect is the result of the discrepancy between the non-linear data which is actually collected and the linear model for the data collection, used in the derivation of the reconstruction algorithms. The algorithm assumes that what is collected are samples of \( Rf^k(t, \omega) \), because of the non-linear nature of the measurement process this is not so. Even if we could measure a projection for all relevant pairs \( (t, \omega) \) our algorithm would not reconstruct \( f^k \) exactly but rather some further non-linear transformation applied to \( f \). In real images the partial volume effect appears as abnormally bright spots or streaks near a hard object, see figure 12.5.
12.2 The point spread function without sampling

Later in this chapter we analyze artifacts which arise in image reconstruction using realistic data and a reasonable model for the source-detector pair. The explanation for a given artifact is usually found by isolating the features of the image which produce it. At the center of this discussion are the point spread and modulation transfer functions (PSF and MTF), characterizing the measurement and reconstruction process. Once the data are sampled the measurement process is no longer translation invariant and therefore it is not described by a single PSF. Instead, for each point \((x, y)\) there is a function \(\Psi(x, y; a, b)\) so
that the reconstructed image at \((x, y)\) is given by

\[ f_\Psi(x, y) = \int_{\mathbb{R}^2} \Psi(x, y; a, b) f(a, b) da db. \]

Because the filter it defines is not shift invariant, strictly speaking \(\Psi\) is not a “point spread function.” Following the standard practice in engineering we also call \(\Psi\) a point spread function, though it must be born in mind that there is a different PSF for each source location \((a, b)\).

Our derivation of \(\Psi\) is done in two steps. First we find a PSF that incorporates a model for the source-detector pair and the filter used in the filtered back-projection step. This part is both translation invariant and isotropic. Afterward we incorporate the effects of sampling the measurements to obtain an expression for \(\Psi(x, y; a, b)\). In this chapter only the parallel beam geometry is considered, our presentation follows [59]. The results for the fan beam geometry are similar but a little more complicated to derive, see [37] and [35]. In this chapter we illustrate the effects of sampling, averaging and various sorts of measurement errors by using a variety of mathematical phantoms as well as real measured data.

### 12.2.1 Point sources

As a function of \((x, y)\), \(\Psi(x, y; a, b)\) is the output of the measurement-reconstruction process applied to a unit point source at \((a, b)\). Mathematically this is modeled by

\[ \delta_{(a,b)}(x, y) = \delta((x, y) - (a, b)). \]

To facilitate the computation of \(\Psi\) it is useful to determine the Radon transform of this generalized function, which should itself be a generalized function on \(\mathbb{R} \times S^1\). Since \(\delta_{(a,b)}\) is \(\delta_{(0,0)}\) translated by \((a, b)\) it suffices to determine \(R\delta_{(0,0)}\). Let \(\varphi_\epsilon\) be a family of smooth functions converging to the \(\delta_{(0,0)}\) in the sense that \(\varphi_\epsilon \ast f\) converges uniformly to \(f\), for \(f\) a continuous function with bounded support. The convolution theorem for the Radon transform, Proposition 6.1.1, says that

\[ R(\varphi_\epsilon \ast f)(t, \omega) = R\varphi_\epsilon \ast Rf(t, \omega). \]

Since the left hand side converges to \(Rf\), as \(\epsilon \to 0\) it follows that

\[ R\delta_{(0,0)}(t, \omega) = \lim_{\epsilon \to 0} R\varphi_\epsilon(t, \omega) = \delta(t). \]

Using Proposition 6.1.2 we obtain the general formula

\[ R\delta_{(a,b)}(t, \omega) = \delta(t - \langle \omega, (a, b) \rangle). \quad (12.4) \]

**Exercise**

**Exercise 12.2.1.** Derive (12.4) by using the family of functions

\[ \varphi_\epsilon(x, y) = \frac{1}{\epsilon^2} \chi_{[0, \epsilon^2]}(x^2 + y^2). \]
12.2. THE PSF

12.2.2 The PSF without sampling

[See: A.4.6.]

For the purposes of this discussion we use the simpler, linear model for a finite width x-ray beam. Let \( w \) be a non-negative function with total integral 1. Our model for a measurement is a sample of

\[
R_W f(t, \omega) = \int_{-\infty}^{\infty} w(u) Rf(t - u, \omega).
\]

If “all” the data

\[ \{R_W f(t, \omega) : t \in [-L, L], \omega \in S^1 \} \]

were available, then the filtered back-projection reconstruction, with filter function \( \phi \), would be

\[
f_{\phi, w}(x, y) = (R^* Q_\phi R_W f)(x, y).
\]

Here \( R^* \) denotes the back-projection operation and

\[
Q_\phi g(t, \omega) = \int_{-\infty}^{\infty} g(t - s, \omega) \phi(s) ds
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{g}(r, \omega) \hat{\phi}(r) e^{irt} dr.
\]

(12.5)

Because \( R_W f \) is defined by convolving \( Rf \) with \( w \) in the \( t \)-parameter, it is a simple computation to see that

\[
f_{\phi, w}(x, y) = R^* Q_{\phi * w} Rf,
\]

(12.6)

where \( \phi * w \) is a 1-dimensional convolution.

Using the central slice theorem in (12.5) gives

\[
f_{\phi, w}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} \int_{0}^{2\pi} \hat{f}(r, \omega) \hat{\phi}(r) \hat{w}(r, \omega) e^{i(x, y) \cdot \omega} r dr d\omega.
\]

(12.7)

As \( \hat{\phi}(r) \approx |r| \) for small \( r \) it is reasonable to assume that \( \hat{\phi}(0) = 0 \) and define \( \hat{\psi}(r) \) by the equation

\[
\hat{\phi}(r) = |r| \hat{\psi}(r).
\]

Substituting this into (12.7) we recognize \( rdrd\omega \) as the area element on \( \mathbb{R}^2 \), to get

\[
f_{\phi, w}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{\psi}(||\xi||) \hat{w}(||\xi||) e^{i(x, y) \cdot \xi} d\xi.
\]

(12.8)

The MTF for the operation \( f \mapsto f_{\phi, w} \) is therefore

\[
\hat{\Psi}_0(\xi) = \hat{\psi}(||\xi||) \hat{w}(||\xi||).
\]

(12.9)
It is important to keep in mind that the Fourier transforms on the right hand side of (12.9) are one-dimensional, while that on the left is a two-dimensional transform.

The PSF is obtained by applying the inverse Fourier transform to the MTF:

$$\Psi_0(x, y) = \frac{1}{|2\pi|^2} \int_{\mathbb{R}^2} \hat{\psi}(\|\xi\|) \hat{w}(\|\xi\|) e^{i(x,y) \cdot \xi} d\xi$$

$$= \frac{1}{2\pi} \int_0^\infty \hat{\psi}(r) \hat{w}(r) J_0(\rho r) r dr,$$

(12.10)

Here $\rho = \|(x, y)\|$ denotes the radius function in the spatial variables. Sampling is not included in this model, however, if $[-B, B]$ is the effective passband of $\hat{\psi}$ then Nyquist’s theorem implies that $d = \frac{\beta}{\alpha}$ is a reasonable proxy for the sample spacing. In the following examples we examine the relationship between the “beam width” and the “sample spacing.” If we replace the beam width, $\delta$, by $\alpha \delta$ and the sample spacing, $d$, by $\beta d$ then the PSF is a positive multiple of

$$\int_0^\infty \hat{\psi}(\frac{\beta}{\alpha} r) \hat{w}(r) J_0(\rho \frac{\beta}{\alpha} r) r dr.$$ 

Hence for a given beam profile and filter function, and up to an overall scaling in $\rho$, the qualitative properties of the PSF depend only on the ratio $\frac{\beta}{\alpha}$. In the examples we use values of $\delta$ and $d$ which are close to 1.

**Example 12.2.1.** For the first example we consider the result of using a sharp cutoff in frequency space. The apodizing function for the filter is $\hat{\psi}(r) = \chi_{[-\frac{\delta}{\alpha}, \frac{\delta}{\alpha}]}(r)$, with the beam profile function

$$w_\delta = \frac{1}{2\delta} \chi_{[-\delta, \delta]}.$$

The MTF is given by

$$\hat{\Psi}_0(\xi) = \text{sinc}(\delta \|\xi\|) \chi_{[-\frac{\delta}{\alpha}, \frac{\delta}{\alpha}]}(\|\xi\|).$$

Figure 12.6(a) shows the PSFs with $d = 1$, $\delta = 0.5, 1$ and 2, figure 12.6(b) shows the corresponding MTFs.
12.2. THE PSF

Figure 12.6: Examples of PSF and MTF with bandlimiting regularization.

Notice the large oscillatory side lobes when the effective sample spacing is greater than the width of the window, i.e. $\delta = .5$. Using a filter of this type may lead to severe Gibbs artifacts, that is, a sharp edge in the original image produces large oscillations, parallel to the edge in the reconstructed image. On the other hand the width of the central peak in the PSF grows as the width of $w$ increases. This is indicative of the lower resolution available in the measured data. With two samples per beam width ($\delta = 2$) the PSF is no longer peaked at zero. In the imaging literature this depression of the PSF near zero is called the volcano effect.

Example 12.2.2. We consider the family of examples with $w_\delta$ as in the previous example and apodizing function

$$\hat{\psi}_\epsilon(r) = e^{-\epsilon|\xi|};$$

the MTF is given by

$$\hat{\psi}_{0(\delta,\epsilon)}(\xi) = \text{sinc}(\delta||\xi||)e^{-\epsilon||\xi||}.$$

If $\epsilon = 0$ (no regularizing function) or $\delta = 0$ (1-dimensional x-ray beam) then the integrals defining $\Psi_0$ exist as improper Riemann integrals,

$$\Psi_{0(\delta,0)}(\rho) = \frac{1}{2\pi\delta} \cdot \frac{\chi(\delta,0)(\rho)}{\sqrt{\delta^2 - \rho^2}}, \quad \Psi_{0(0,\epsilon)}(\rho) = \frac{\epsilon}{2\pi} \cdot \frac{1}{[\epsilon^2 + \rho^2]^{\frac{1}{2}}}.$$

The graphs of these functions are shown figure 12.7(a), the dotted curve shows $\Psi_{0(5,0)}$ and the solid line is $\Psi_{0(0,5)}$. Figure 12.7(b) shows the corresponding MTFs. The PSF in the $\epsilon = 0$ case displays an extreme version of the volcano effect.
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Figure 12.7: Limits for the PSF and MTF in the filtered back-projection algorithm.

Graphs of $\Psi_{0(\delta,\epsilon)}(\rho)$, for several values of $(\delta, \epsilon)$ are shown in figure 12.8(a). The MTFs are shown in figure 12.8(b). The values used are $(.25, .05), (.125, .05), (.125, .125), (.125, .3))$, smaller values of $\epsilon$ producing a sharper peak in the PSF. For $\epsilon \ll \delta$ the PSF resembles the limiting case, $\epsilon = 0$, with a crater near $\rho = 0$.

Figure 12.8: Examples of PSF and MTF with exponential regularization.
12.2. THE PSF

Figure 12.9: Examples of PSF and MTF with Shepp-Logan regularization.

Example 12.2.3. As a final example we consider the Shepp-Logan filter. The regularizing filter has (one-dimensional) transfer function

\[ \hat{\phi}(r) = |r| \left| \text{sinc} \left( \frac{dr}{2} \right) \right|^3 \]

Using the same model for the source-detector pair as before gives the total MTF:

\[ \Psi_{0,(\delta,d)}(\xi) = \text{sinc}(\delta||\xi||) \left| \text{sinc} \left( \frac{d||\xi||}{2} \right) \right|^3. \]

Recall that the Shepp-Logan filter is linearly interpolated and \( d \) represents the sample spacing. Here \( 2\delta \) is the width of the source-detector pair. Graphs, in the radial variable of the PSFs and corresponding MTFs, for the pairs \((.125,.05), (.125,.125), (.125,.3)\) are shown in figure 12.9. Again smaller values of \( d \) produce a more sharply peaked PSF.

It is apparent in the graphs of the PSFs with exponential and Shepp-Logan regularization that these functions do not have long oscillatory tails and so the effect of convolving a piecewise continuous, bounded function with \( \Psi_0 \) should be an overall blurring, without oscillatory artifacts. They are absent because the MTF decays smoothly and sufficiently rapidly to zero. The PSFs obtained using a sharp cut-off in frequency have long oscillatory tails which, in turn produce Gibbs artifacts in the reconstructed images. Oscillatory artifacts can also result from sampling. This is considered in the following section. From both (12.10) and (12.11) it is clear that the roles of the beam profile \( w \) and the filter function \( \phi \) are entirely interchangeable in the unsampled PSF. This is no longer the case after sampling is done in the \( t \)-parameter.

These examples all indicate that, once the beam width is fixed, the full width half maximum of the PSF is not very sensitive to the sample spacing. However, smaller sample spacing produces a sharper peak which should in turn lead to less blurring in the reconstructed image. From the limiting case shown in figure 12.7(a) it is clear that the resolution is ultimately limited by the beam width. Since the PSF tends to infinity the FWHM definition of resolution is not applicable. Half of the volume under the PSF (as a radial function
on \( \mathbb{R}^2 \) lies in the disk of radius \( d/2 \), indicating that the maximum available resolution, with the given beam profile is about half the width of the beam. This is in good agreement with experimental results that show that having 2 samples per beam width leads to a better reconstruction, though little improvement is seen beyond 4 samples per beam width, see [59] or [39]. To measure the resolution of a CT-machine or reconstruction algorithm it is customary to use a “resolution phantom.” This is an array of disks of various sizes with various spacings. An example is shown in figure 12.10.

![Resolution Phantom and Reconstruction](image)

(a) A resolution phantom.  
(b) Its reconstruction using a fan beam algorithm.

Figure 12.10: Resolution phantoms are used to measure the actual resolution of a CT-scanner and/or reconstruction algorithm.

**Exercises**

**Exercise 12.2.2.** Using the formula for \( \Psi_0(x, y) \) derive the alternate expression for \( \Psi_0(x, y) \):

\[
\Psi_0(x, y) = \frac{1}{2\pi} \int_0^\pi \int_{-\infty}^\infty w((x, y), \omega - s)\phi(s) ds d\omega.
\]

**Exercise 12.2.3.** By considering the decay properties of the MTFs, in examples 12.2.2 and 12.2.3 explain why one does not expect the PSFs to have slowly decaying, oscillatory tails.

**Exercise 12.2.4.** Using the graphs in the examples above determine the FWHM of each of the PSFs for which it makes sense. For which cases is the FWHM definition of resolution inapplicable? What definition of resolution would be more meaningful in these cases?
12.3 The PSF with sampling

(a) Reconstruction using parallel beam data.  
(b) Reconstruction using fan beam data.

Figure 12.11: Reconstructions of the Shepp-Logan phantom using filtered back-projection algorithms.

Real measurements entail both ray and view sampling. For a parallel beam machine, ray sampling refers to sampling in the $t$ parameter and view sampling to the $\omega$ (or $\theta$) parameter. For the sake of simplicity these effects are usually considered separately. We follow this procedure, first finding the kernel function incorporating ray sampling and then view sampling. Each produces distinct artifacts in the reconstructed image. As ray sampling is not a shift invariant operation, the measurement and reconstruction process can no longer be described by a single PSF, but instead requires a different integrand for each point in the reconstruction grid. For the purpose of comparison, the PSFs are often evaluated at the center of reconstruction grid (i.e. $(0,0)$), though it is also interesting to understand how certain artifacts depend on the location of the input. In the previous section we obtained the PSF for unsampled data, with a reasonable filter function it was seen to produce an overall blurring of the image, without oscillatory effects. Both aliasing artifacts and the Gibbs phenomenon are consequences of slow decay in the Fourier transform which is typical of functions that change abruptly. One therefore would expect to see a lot of oscillatory artifacts produced by inputs with sharp edges. To test algorithms one typically uses the characteristic functions of disks or polygons placed at various locations in the image. Figure 12.11 shows reconstructions of the Shepp-Logan phantom, figure 11.3 made with filtered back-projection algorithms. Note the oscillatory artifacts parallel to sharp boundaries as well as the patterns of oscillations in the exterior region.

12.3.1 Ray sampling

Suppose that $d$ is the sample spacing in the $t$-parameter and that the image is reconstructed using a Ram-Lak or linearly interpolated filter. The choice of interpolation scheme is important because it affects the rate of decay for the MTF, see the appendix to section 11.3.4.
For the purposes of this paragraph we suppose that sampling is only done in the affine parameter so that
\[ \{ R_W f(jd, \omega) : j = -N, \ldots, N \} \]
is collected for all \( \omega \in S^1 \). With \( \phi \) the filter function, the reconstructed image is
\[ \tilde{f}_{\phi,w}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} Q_{\phi,w} \tilde{f}(\langle (x, y), \omega \rangle, \omega) d\omega, \quad (12.12) \]
where
\[ Q_{\phi,w} \tilde{f}(t, \omega) = d \sum_{j=-\infty}^{\infty} \phi(t - jd) R_W f(jd, \omega). \]
For the derivation of the PSF we let \( f = \delta_{(a,b)} \), as noted above
\[ R_W f(jd, \omega) = w(jd - \langle (a, b), \omega \rangle). \]
The linear interpolation, used in a Ram-Lak filter, is easiest to incorporate in the Fourier representation. If
\[ \hat{\phi}_p(r) = \sum_{j=-\infty}^{\infty} \phi(jd)e^{-ijd}r \]
then, as shown in section 11.3.4,
\[ \hat{\phi}(r) = \text{sinc}^2 \left( \frac{rd}{2} \right) \hat{\phi}_p(r). \]
The Fourier transform of \( Q_{\phi,w} \tilde{f} \) in the \( t \)-variable is given by
\[ \widehat{Q_{\phi,w} \tilde{f}}(r, \omega) = d \text{sinc}^2 \left( \frac{rd}{2} \right) \hat{\phi}_p(r) \sum_{j=-\infty}^{\infty} w(jd - \langle (a, b), \omega \rangle) e^{-ijd}r. \quad (12.13) \]
To evaluate the last sum we can use the dual Poisson summation formula, (8.13), obtaining
\[ d \sum_{j=-\infty}^{\infty} w(jd - \langle (a, b), \omega \rangle) e^{-ijd}r = e^{-i\langle (a, b), r\omega \rangle} \sum_{j=-\infty}^{\infty} \hat{w}(r + \frac{2\pi j}{d}) e^{-i\frac{2\pi j}{d} \langle (a, b), \omega \rangle}, \quad (12.14) \]
For this computation to be valid we need to assume that both \( w \) and \( \hat{w} \) decay sufficiently rapidly for the Poisson summation formula to be applicable. In particular \( w \) must be smoother than the functions, \( w_\delta \), used in example 12.2.2.
Using the Fourier inversion formula to express \( Q_{\phi,w} \tilde{f} \) in (12.12) gives the kernel function,
\[ \Psi(x, y; a, b) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \text{sinc}^2 \left( \frac{rd}{2} \right) \hat{\phi}_p(r) e^{i\langle (x-a, y-b), r\omega \rangle} \times \]
\[ \left[ \sum_{j=-\infty}^{\infty} \hat{w}(r + \frac{2\pi j}{d}) e^{-i\frac{2\pi j}{d} \langle (a, b), \omega \rangle} \right] drd\omega. \quad (12.15) \]
12.3. THE PSF WITH SAMPLING

It is quite apparent that $\Psi$ is not a function of $(x - a, y - b)$. The symmetry in the roles played by $\phi$ and $w$ has also been lost. The infinite sum in (12.15) leads to aliasing errors, a sharper beam profile producing larger errors. This infinite sum defines a $\frac{2\pi}{d}$-periodic function of $r$. In terms of the rate of decay, the principal difference between the integrand in (12.10) and that in (12.15) is that the decay coming from $\hat{\omega}(||\xi||)$ is lost.

**Example 12.3.1.** We first consider simple bandlimiting with the regularizing filter defined by $\hat{\psi}_d(r) = \chi[-\frac{\pi}{d}; \frac{\pi}{d}](r)$, and the rectangular windows defined by $w_d$. In this case we can find an explicit formula for $\hat{\phi}_p$. At the sample points we have

$$\phi_d(jd) = \begin{cases} \frac{\pi}{2d^2} & \text{if } j = 0, \\ \frac{(-1)^j}{\pi j^2 d^2} & \text{if } j \neq 0. \end{cases}$$

From the definition it follows that

$$\hat{\phi}_p(r) = \frac{\pi}{2d^2} - \frac{4}{\pi d^2} \sum_{j=1}^{\infty} \frac{\cos(2j - 1)dr}{(2j - 1)^2}. \quad (12.16)$$

The function on the right hand side of (12.16) has a simple formula: it is a $\frac{2\pi}{d}$-periodic function with:

$$\hat{\phi}_p(r) = \frac{|r|}{d} \text{ for } |r| < \frac{\pi}{d}. \quad (12.17)$$

Though the functions, $w_\delta$, are too singular to apply the Poisson summation argument, the sums on the left hand side of (12.14) are finite. The MTF for this combination of filtering, sample spacing and beam profile is

$$\Psi_{\delta,d}(\xi; a, b) = d \sin^2 \left( \frac{||\xi||d}{2} \right) \hat{\phi}_p(||\xi||)||\xi||^{-1} \times \sum_{j=-\infty}^{\infty} w_d(jd - (a, b), \frac{\xi}{||\xi||})e^{-ijd||\xi||}. \quad (12.18)$$

For example if $d = 1$, $\delta = .5$ and $(a, b) = (0, 0)$ then only the $j = 0$ term is non-zero and this reduces to

$$\Psi_{.5,1}(\xi; 0, 0) = \sin^2 \left( \frac{||\xi||}{2} \right) \hat{\phi}_p(||\xi||)||\xi||^{-1}.$$
The graphs in 12.12 should be compared with the unsampled versions in figure 12.6. Somewhat paradoxically the sidelobes of the sampled PSF are not much larger than the unsampled. This is because the linear interpolation used to define \( \phi \) between sample points leads to a smoother MTF which decays like \( |\xi|^{-3} \). For \( \delta = .5 \) the sidelobes in the sampled PSF are smaller! In this case the unsampled MTF has a jump discontinuity at \( \pi \) which explains the sinc-like behavior in the PSF. The linear interpolation is also evident in the graphs in 12.12(a). As the beam width increases (hence: more samples per beam width) the peak of the PSF broadens and it displays a more pronounced volcano effect.

We conclude this example with contour plots of \( \Psi_{1,1}(x, y; a, b) \) for several values of \((a, b)\). The plots in figure 12.13 were computed using a finite approximation to the inverse Fourier transform and rectangular partial sums. Figure 12.13(a) is a contour plot of \( \Psi_{1,1}(x, y; 0, 0) \). Near to the principal peak the contours are circles, closer to the boundary of the plot the radial symmetry starts to break down. This is a consequence of the rectangular partial sums used in the numerical computation. Such sums are used in actual reconstructions and one also sees artifacts with a similar rectangular symmetry. These artifacts are not caused by noise, measurement error or even by sampling, *per se* but rather by the precise nature of the approximation to the inverse Fourier transform used in the numerical algorithms. Figures 12.13(b-d) show \( \Psi_{1,1}(x, y; a, b) \) for three other values of \((a, b)\). Each plot shows a very pronounced peak centered at \((a, b)\). As none of these plots displays even an approximate radial symmetry, it is clear that they are not the result of translating \( \Psi_{1,1}(x, y; 0, 0) \). Note finally that the Gibbs oscillation is most pronounced for \((a, b) = (0, 0)\).
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Figure 12.13: Contour plots of the ray sampled kernel function using a rectangular beam profile, and simple bandlimiting with linear interpolation. These plots show that once sampling is done in the affine parameter the measurement-reconstruction process is no longer shift invariant.

Example 12.3.2. In this example we use $w_\delta$ convolved with a Gaussian. The window function $w_\delta$ models the detector while the Gaussian models the x-ray source. A Gaussian focal spot of “width” $h/\sqrt{2}$ is described by the function:

$$s_h(u) = \frac{1}{\pi h^2} e^{-\left(\frac{u}{h}\right)^2}.$$

For our examples we fix $h = \frac{1}{2}$ and $\delta = 1$. The total beam profile is then

$$w(u) = \frac{1}{2} \int_{-1}^{1} s_1(u-v) dv,$$
with
\[ \hat{w}(r) = \text{sinc}(r)e^{-\left(\frac{r^2}{16}\right)}. \]

We use the Shepp-Logan regularizing filter, for which
\[ \hat{\phi}_\rho(r) = |r \cdot \text{sinc} \left( \frac{rd}{2} \right)|. \]

As before, the issue here is the relationship between \( d \), the sample spacing and the “width” of \( w \). Colloquially one asks for the “number of samples per beam width.” With the given parameters, the FWHM(\( w \)) is very close to 2.

While the overall filtering operation is no longer isotropic, the function \( \Psi(x, y; 0, 0) \) is radial; figures (12.14)(a,b,c) show this function (of \( \rho \)) with various choices for \( d \), with and without the effects of ray sampling. The dotted line is the unaliased PSF and the solid line the aliased. As before, smaller values of \( d \) give rise to sharper peaks in the PSF. The corresponding MTFs are shown in figure 12.14(d).

![Figure 12.14: The effect of ray sampling on the PSF.](image-url)
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The graphs on the right hand side of 12.14(d) include the effect of aliasing while those on the left are the unaliased MTFs, as $d$ decreases, the “passband” of the MTF broadens. With this choice of beam profile and regularizing filter, once there is at least one sample per beam width, the resolution, measured by FWHM, is not affected very much by aliasing. Though it is not evident in the pictures, these PSFs also have long oscillatory tails. The very small amplitude of these tails is a result of using a smooth, rapidly decaying regularizing function.

Exercises

Exercise 12.3.1. When doing numerical computations of $\Psi$ it is sometimes helpful to use the fact that

$$
\int_{-\infty}^{\infty} w(jd - ((a, b), \omega))e^{-ijdr}
$$

is a periodic function. Explain this observation and describe how it might figure in a practical computation. It might be helpful to try to approximately compute this sum using both representations in (12.14).

Exercise 12.3.2. Verify (12.17) by a direct computation.

Exercise 12.3.3. Continue the computations begun in example 12.3.1 and draw plots of $\Psi_{a,d}(x, y; a, b)$ for other values of $(a, b)$ and $(\delta, d)$. Note that $\Psi_{a,d}$ is no longer a radial function of $(x, y)$, so a two-dimensional plot is required. Repeat this experiment with the beam profile and apodizing function used in example 12.3.2.

12.3.2 View sampling

We now turn to artifacts which result from using finitely many views and begin by considering the reconstruction of a mathematical phantom made out of constant density elliptical regions. In figure 12.15 note the pattern of oscillations in the exterior region along lines, tangent to the boundary of ellipse and the absence of such oscillations in the interior. A somewhat subtler observation is that the very pronounced, coherent pattern of oscillations does not begin immediately but rather at a definite distance from the boundary of the ellipse. This phenomenon is a consequence of sampling in the angular parameter and the filtering operations needed to approximately invert the Radon transform. Our discussion of these examples closely follows that in [72].
Example 12.3.3. Suppose the object, $E$ of constant density 1 with boundary the locus of points, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The line integral of $f = \chi_E$ along a line $l_{t,\theta}$ in simply the length of the intersection of the line with $E$. Let $s_{\pm}(t, \theta)$ denote the $s$-parameters for the intersection points of the line $l_{t,\theta}$ with the boundary of $E$. The distance between these two points $|s_+(t, \theta) - s_-(t, \theta)|$ is $Rf(t, \omega(\theta))$. Plugging the parametric form of the line into the equation for the ellipse and expanding gives

$$s^2\left(\frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2}\right) + 2st \sin \theta \cos \theta \left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{t^2 \cos^2 \theta}{a^2} + \frac{t^2 \sin^2 \theta}{b^2} - 1 = 0.$$ 

Re-write the equation as

$$p(t, \theta)s^2 + q(t, \theta)s + r(t, \theta) = 0,$$

$p, q$ and $r$ are the corresponding coefficients. The two roots are given by

$$s_\pm = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p},$$

the distance between the two roots is therefore

$$s_+ - s_- = \frac{\sqrt{q^2 - 4pr}}{p}.$$

This gives the formula for $Rf$:

$$Rf(t, \omega(\theta)) = \begin{cases} 
2\beta(\theta)\sqrt{\alpha(\theta)^2 - t^2} & |t| \leq \alpha(\theta), \\
0 & |t| > \alpha(\theta),
\end{cases}$$

where

$$\alpha(\theta) = \sqrt{\frac{a^4 \cos^2(\theta) + b^4 \sin^2(\theta)}{a^2 \cos^2(\theta) + b^2 \sin^2(\theta)}},$$

$$\beta(\theta) = \sqrt{\frac{(a^2 \cos^2(\theta) + b^2 \sin^2(\theta))(b^2 \cos^2(\theta) + a^2 \sin^2(\theta))}{a^4 \cos^2(\theta) + b^4 \sin^2(\theta)}}.$$
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Both $\alpha$ and $\beta$ are smooth, non-vanishing functions of $\theta$.

Doing the exact filtration,

$$\mathcal{G} f(t, \omega(\theta)) = \frac{1}{k} \mathcal{H} \partial_t R f(t, \omega(\theta)),$$

gives

$$\mathcal{G} f(t, \omega(\theta)) = \begin{cases} 2\beta(\theta) & |t| \leq \alpha(\theta), \\ 2\beta(\theta)(1 - \frac{|t|}{\sqrt{t^2 - \alpha^2(\theta)}}) & |t| > \alpha(\theta). \end{cases}$$

In an actual reconstruction this filter is regularized. Let $Q\phi f$ denote the approximately filtered measurements, with filter function $\phi$. Approximating the back-projection with a Riemann sum gives

$$\tilde{f}_\phi(x, y) \approx \frac{1}{2(M + 1)} \sum_{k=0}^{M} Q\phi f((x, y), \omega(k\Delta\theta), \omega(k\Delta\theta)).$$

For points inside the ellipse

$$\tilde{f}_\phi(x, y) \approx \frac{1}{2(M + 1)} \sum_{k=0}^{M} \beta(k\Delta\theta).$$

This is well approximated by the Riemann sum because $\beta$ is a smooth bounded function. For the points outside the ellipse there are three types of lines which appear in the back-projection:

1. Lines which pass through $E$,
2. Lines which are distant from $E$,
3. Lines outside $E$ which pass very close to the boundary of $E$. 

Figure 12.16: Parameters describing the Radon transform of $\chi_E$. 
The first two types of lines are not problematic. However for any point in the exterior of the ellipse, the back-projection involves lines which are exterior to the ellipse but pass very close to it. This leads to the oscillations apparent in the reconstruction along lines tangent to the boundary of the regions. This is a combination of the Gibbs phenomenon and aliasing. To compute an accurate value for $\tilde{f}_\varphi$ at an exterior point requires delicate cancellations between the moderate positive and sometimes large negative values assumed by $Q_\varphi f$. For points near enough to the boundary of $E$ there is a sufficient density of samples to obtain the needed cancellations. For more distant points the cancellation does not occur and the pattern of oscillations appears. In the next paragraph we derive a “far field” approximation to the reconstruction of such a phantom. This gives, among other things, a formula for the radius where the oscillatory artifacts first appear. In [37] such a formula is derived for a fan beam scanner, using their approach we derive a similar formula for the parallel beam case. Also apparent in figure 12.15 is an oscillation very near and parallel to the boundary of $E$; this is the usual combination of the Gibbs phenomenon and aliasing caused by ray sampling.

**Example 12.3.4.** An very striking example of this phenomenon can be seen in the reconstruction of a rectangular region. If $f$ is the characteristic function of the square with vertices $\{(-1, -1)\}$ then for $|\theta| < \frac{\pi}{4}$

$$R f(t, \omega(\theta)) = \begin{cases} 0 & \text{if } |t| > \cos \theta + \sin \theta, \\ \frac{\cos \theta + \sin \theta - t}{\cos \theta \sin \theta} & \text{if } \cos \theta - \sin \theta < t < \cos \theta + \sin \theta, \\ \frac{\cos \theta}{\cos \theta + \sin \theta + t} & \text{if } \sin \theta - \cos \theta < t < \cos \theta - \sin \theta, \\ - \frac{\cos \theta + \sin \theta}{\cos \theta \sin \theta} & \text{if } - (\cos \theta + \sin \theta) < t < \sin \theta - \cos \theta. \end{cases}$$

The function $R f(t, \omega(\theta))$ is periodic of period $\frac{\pi}{2}$ in the $\theta$ parameter. If $\theta \neq 0$ then $R f(t, \theta)$ is a continuous, piecewise differentiable function in $\theta$, whereas

$$R f(t, (1, 0)) = \chi_{[-2,2]}(t)$$

has a jump discontinuity. Note the pronounced oscillatory artifact in the exterior of the square along lines tangent to the sides of the square in figure 12.17. As before there is also a Gibbs oscillation in the reconstructed image, parallel to the boundary of the square.
**Exercises**

**Exercise 12.3.4.** Derive the formula for $Rf$ and $Gf$ in example 12.3.3.

**Exercise 12.3.5.** Compute $-i\mathcal{H}\partial_t Rf$ for example 12.3.4.

**A far field approximation for the reconstructed image**

In this paragraph we obtain an approximate formula for the Ram-Lak reconstruction of a small radially symmetric object at points far from the object. Let $\phi$ denote a Ram-Lak filter function and $w$ a function describing the source-detector pair. If $f$ is the data then the approximate, filtered back-projection reconstruction is given by

$$f_{\phi,w}(x,y) = \frac{1}{2\pi} \int_0^\pi Q_{\phi,w}f((x,y),\omega)d\omega,$$

where

$$Q_{\phi,w}(t,\omega) = \int_{-\infty}^\infty R_W f(s,\omega)\phi(t-s)ds.$$

Here $R_W f$ denotes the “$w$-averaged Radon transform” of $f$. We now consider the effect of sampling in the $\omega$-parameter, leaving $t$ as a continuous parameter. Equation (12.6) shows that, in this case, $\phi$ and $w$ are interchangeable; the effects of finite beam width and regularizing the filter are both captured by using $\phi \ast w$ as the filter function. We analyze the difference between $f_{\phi,w}(x,y)$ and the Riemann sum approximation

$$\tilde{f}_{\phi,w}(x,y) = \frac{1}{4\pi} \sum_{j=0}^M Q_{\phi,w}f((x,y),\omega(j\Delta\theta)), \omega(j\Delta\theta))\Delta\theta.$$

For simplicity we restrict attention to functions of the form

$$f^{(a,b)}(x,y) = f((x,y) - (a,b)),$$

where $f$ is a radial function. The averaged Radon transform of $f$, is independent of $\omega$; to simplify the notation we suppress the $\omega$-dependence writing

$$R_W f(t), \quad Q_{\phi,w}f(t)$$

and therefore

$$R_W f^{(a,b)}(t,\omega) = R_W f(t - ((a,b),\omega)), \quad Q_{\phi,w}f^{(a,b)}(t,\omega) = Q_{\phi,w}f(t - ((a,b),\omega)), \quad (12.19)$$

as well. From equation (12.19) we obtain

$$f_{\phi,w}^{(a,b)}(x,y) = \frac{1}{2\pi} \int_0^\pi Q_{\phi,w}f((x,y) - (a,b),\omega))d\omega.$$
Letting \((x - a, y - b) = R(\cos \varphi, \sin \varphi)\) and \(\omega(\theta) = (\cos \theta, \sin \theta)\) gives

\[
f_{\phi, w}^{(a,b)}(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} Q_{\phi, w} f(R \cos(\theta - \varphi)) d\theta
\]

with Riemann sum approximation:

\[
\tilde{f}_{\phi, w}^{(a,b)}(x, y) = \frac{1}{2\pi} \sum_{j=0}^{M} Q_{\phi, w} f(R \cos(j\Delta\theta - \varphi)) \Delta \theta.
\] (12.20)

The objects of principal interest in this analysis are small and hard. From the examples presented in the previous paragraph, our primary interest is the reconstruction at points in the exterior of the object. The function \(f_{\phi, w}^{(a,b)}\) is an approximation to \(f * k\), where \(k\) is the inverse Radon transform of \(w\). If \(w\) has bounded support and \(\phi\) provides a good approximation to \(-i\phi(t)\), then \(f_{\phi, w}^{(a,b)}\) should be very close to zero for points outside the support of \(f * k\). Indeed, if \(w\) has small support then so does \(k\) and therefore the support of \(f * k\) is a small enlargement of the support of \(f\) itself. We henceforth assume that, for points of interest,

\[
f_{\phi, w}^{(a,b)}(x, y) \approx 0
\] (12.21)

and therefore any significant deviation of \(\tilde{f}_{\phi, w}^{(a,b)}(x, y)\) from zero is an error.

If \(f\) is the characteristic function of a disk of radius \(r\) then \(Q_{\phi, w} f(t)\) falls off rapidly for \(|t| >> r\), see example 11.3.2. There is a \(j_0\) so that

\[
j_0 \Delta \theta - \varphi < \frac{\pi}{2},
\]

\[
(j_0 + 1) \Delta \theta - \varphi \geq \frac{\pi}{2}.
\] (12.22)

If we let \(\Delta \varphi = \frac{\pi}{2} - j_0 \Delta \theta + \varphi\) then \(0 < \Delta \varphi \leq \Delta \theta\) and

\[
\tilde{f}_{\phi, w}^{(a,b)}(x, y) = \frac{1}{2\pi} \sum_{j=0}^{M} Q_{\phi, w} f(R \sin(j\Delta\theta - \Delta \varphi)) \Delta \theta.
\]

As the important terms in this sum are those with \(|j|\) close to zero, we approximate it by using

\[
\sin(j\Delta\theta - \Delta \varphi) \approx j\Delta\theta - \Delta \varphi
\]

obtaining,

\[
\tilde{f}_{\phi, w}^{(a,b)}(x, y) \approx \frac{1}{2\pi} \sum_{j=\infty}^{\infty} Q_{\phi, w} f(R(j\Delta\theta - \Delta \varphi)) \Delta \theta.
\]

The limits of summation have also been extended from \(-\infty\) to \(\infty\). The error this introduces is small as \(\phi(t) = O(t^{-2})\).

The Poisson summation formula can be used to evaluate the last expression, it gives

\[
\tilde{f}_{\phi, w}^{(a,b)}(x, y) \approx \frac{1}{2\pi R} \sum_{j=\infty}^{\infty} \widetilde{Q_{\phi, w}} \left( \frac{2\pi j}{R\Delta \varphi} \right) e^{-\frac{2\pi j(\Delta \varphi)}{\Delta \theta}}.
\]
From the central slice theorem
\[ \tilde{Q}_{\phi,w} f(\rho) = \hat{\phi}(\rho) \hat{\hat{w}}(\rho) \hat{\hat{f}}(\rho). \]

Assuming the \( \hat{\phi}(0) = 0 \) and that \( w \) is an even function gives the simpler formula
\[
\tilde{f}^{(a,b)}_{\phi,w}(x,y) \approx \frac{1}{\pi R} \sum_{j=1}^{\infty} \hat{\phi} \cdot \hat{\hat{w}} \cdot \hat{\hat{f}} \left( \frac{2\pi j}{R \Delta \theta} \right) \cos \left( \frac{2\pi j \Delta \varphi}{\Delta \theta} \right). \tag{12.23}
\]

In order for this sum to be negligible at a point whose distance to \( (a,b) \) is \( R \), the angular sample spacing, \( \Delta \theta \) must be chosen so that the effective support of \( \hat{\phi} \cdot \hat{\hat{w}} \cdot \hat{\hat{f}} \) is contained in
\[
\left( -\frac{2\pi}{R \Delta \theta}, \frac{2\pi}{R \Delta \theta} \right).
\]

This explains why the oscillatory artifacts only appear at points that are at a definite distance from the object: for small values of \( R \) the sum itself is very small. For large enough \( R \) the product \( \hat{\phi} \cdot \hat{\hat{w}} \cdot \hat{\hat{f}} \) is evaluated at small arguments and the sum may become large.

Suppose, for example that \( R \Delta \theta \) is such that all terms in this sum but the first are negligible, then
\[
\tilde{f}^{(a,b)}_{\phi,w}(x,y) \approx \frac{1}{\pi R} \hat{\phi} \cdot \hat{\hat{w}} \cdot \hat{\hat{f}} \left( \frac{2\pi}{\| (x,y) - (a,b) \| \Delta \theta} \right) \cos \left( \frac{2\pi \Delta \varphi}{\Delta \theta} \right). \tag{12.24}
\]

The cosine factor produces an oscillation in the sign of the artifact whose period equals \( \Delta \theta \). This is quite apparent in figures 12.19 and 12.18. The amplitude of the artifact depends on the distance to the object through the product \( \hat{\phi} \cdot \hat{\hat{w}} \cdot \hat{\hat{f}} \). This allows us to relate the angular sample spacing, needed to obtain an artifact free reconstruction in a disk of given radius, to the source-detector function \( w \). For simplicity suppose that \( w(u) = \frac{1}{2\delta} \chi_{[-\delta,\delta]}(u) \) so that \( \hat{\hat{w}}(\rho) = \text{sinc}(\rho \delta) \).

The first zero of \( \hat{\hat{w}} \) occurs at \( \rho = \pm \frac{\pi}{\delta} \) which suggests that taking
\[
\Delta \theta < \frac{2\delta}{R}
\]
is a minimal requirement to get “artifact free” reconstructions in the disk of radius \( R \). This ignores the possible additional attenuation of the high frequencies resulting from \( \hat{\phi} \); which is consistent with our desire to get a result that is independent of the sample spacing in the \( t \)-parameter. The estimate for \( \Delta \theta \) can be re-written
\[
\frac{\pi R}{2\delta} < \frac{\pi}{\Delta \theta}.
\]

The quantity on the right hand side is the number of samples, \( M + 1 \) in the \( \omega \) direction. As \( 2\delta \) is the width of the source, Nyquist’s theorem implies that the maximum spatial frequency available in the data is about \( (4\delta)^{-1} \). If we denote this by \( \nu \) then the estimate reads
\[
2\pi R \nu < M + 1.
\]
Essentially the same result was obtained in section 11.3.5, with much less effort! The difference in the analyses is that, in the earlier discussion, it was assumed that the data is essentially bandlimited to \([\pi/2^\omega, \pi/4^\omega]\). Here this bandlimiting is a consequence of the low pass filtering which results from averaging over the width of the x-ray beam.

It is important to note that the artifacts which result from view sampling are present whether or not the data is sampled in the \(t\)-parameter. These artifacts can be reduced by making either \(\phi\) or \(w\) smoother. This is in marked contrast to the result obtained for ray sampling. In that case the aliasing errors are governed solely by \(w\) and cannot be reduced by changing \(\phi\). If \(f\) describes a smooth object, so that \(\hat{f}\) decays rapidly, then it is unlikely that view sampling aliasing artifacts will appear in the reconstruction region.

**Example 12.3.5.** To better understand formula (12.24) we consider \((a,b) = (0,0)\), and \(f(x,y) = \chi_{D_{1/4}}(x,y)\). For simplicity we use \(w = \frac{1}{2}\chi_{[-1,1]}\) for the beam profile and the Shepp-Logan filter with \(d = .25\). We consider the right hand side of 12.24 with \(\Delta\theta = \frac{2\pi}{8}\) and \(\frac{2\pi}{32}\). The 3-dimensional plots in figures 12.18(a) and 12.19(b) of the functions defined in (12.20) give an idea of how the artifacts appear in a reconstructed image. Notice that the sign of the error reverses along a circle. The 2-dimensional graphs in these figures are sections of the 3-dimensional plots along lines of constant \(\varphi\). These graphs allow for a quantitative appreciation for the size of the errors and their dependence on \(\Delta\theta\).
12.3. THE PSF WITH SAMPLING

(a) 3d-plot. (b) Sections along radii.

Figure 12.19: View sampling artifacts with $\Delta \theta = \frac{2\pi}{8}$

Remark 12.3.1. In [37] a similar analysis is presented for a fan beam machine. The results are similar though a bit more complicated. The artifacts produced by view sampling in a fan beam machine differ in one important way: In a parallel beam machine the pattern of oscillations is circularly symmetric and depends on the distance from the center of a radially symmetric object. For a fan beam machine the pattern displays a similar circular symmetry but the center of the circle no longer agrees, in general, with the center of the object, see figure 12.20.

(a) Parallel beam. (b) Fan beam.

Figure 12.20: Examples comparing view aliasing in parallel beam and fan beam scanners. The view sampling artifact is the pattern of light and dark radial lines beginning about four diameters from the circle.

Exercises
Exercise 12.3.6. Given that we use the Poisson summation formula to derive (12.23), why is it allowable to use $w(u) = (2\delta)^{-1}\chi_{[\delta,\delta]}(u)$ in this analysis.

Exercise 12.3.7. Show that the PSF for a Ram-Lak reconstruction, incorporating the beam width function $w$ and sampling in the $\omega$-parameter is

$$\Psi(x, y; a, b) = \frac{\Delta \theta}{2\pi} \sum_{j=0}^{M} \phi \ast w((x - a, y - b), \omega(j\Delta \theta)).$$

Note that this operation is shift invariant.

Exercise 12.3.8. Using the result of the previous exercise, compute the MTF for a Ram-Lak reconstruction incorporating the beam width function $w$ and sampling in the $\omega$-parameter.

12.4 The effects of measurement errors

The artifacts considered in the previous sections are algorithmic artifacts, resulting from the sampling and approximation used in any practical reconstruction method. The final class of linear artifacts we consider are the effects of systematic measurement errors. This should be contrasted to the analysis in Chapter 16 of the effects of random measurement errors or noise. Recall that the measurements made by a CT-machine are grouped into views and each view is comprised of a collection of rays. We now consider the consequences of having a bad ray in a single view, a bad ray in every view and a single bad view. These analyses illustrate a meta-principle, called the smoothness principle, often invoked in medical imaging,

The filtered back-projection algorithm is very sensitive to errors that vary abruptly from ray to ray or view to view but relatively tolerant of errors that vary gradually.

This feature of the algorithm is a reflection of the fact that the filter function, $\hat{\phi}(r)$ approximates $|r|$ and therefore attenuates low frequencies and amplifies high frequencies. Our discussion is adapted from [73], [35] and [36]. Many other errors are analyzed in these references.

For this analysis we suppose that the measurements, made with a parallel beam scanner, are the samples

$$\{P(t_j, \omega(k\Delta \theta)) : k = 0, \ldots, M, j = 1, \ldots, N\}$$

of $R_W f(t, \omega)$. The coordinates are normalized so that the object lies in $[-1, 1] \times [-1, 1]$. The angular sample spacing is

$$\Delta \theta = \frac{\pi}{M + 1}$$

and the rays are uniformly sampled at

$$t_j = -1 + (j - \frac{1}{2})d.$$ 

If $\phi$ is the filter function, which is specified at the sample points and linearly interpolated in between, then the approximate reconstruction is given by

$$\tilde{f}_\phi(x, y) = \frac{1}{N(M + 1)} \sum_{k=0}^{M} \sum_{j=1}^{N} P(t_j, \omega(k\Delta \theta)) \phi(x \cos(k\Delta \theta) + y \sin(k\Delta \theta) - t_j).$$ (12.25)
For the purposes of comparison we use the Shepp-Logan filter

\[ \phi(0) = \frac{4}{\pi d^2}, \quad \phi(jd) = \frac{-4}{\pi d^2(4jd - 1)}. \]

### 12.4.1 A single bad ray

The first effect we consider is an isolated measurement error in a single ray, from a single view. Suppose that \( P(t_{j_0}, \omega(k_0 \Delta \theta)) \) differs from the “true” value by \( \epsilon \). As formula (12.25) is linear, this measurement error produces a reconstruction error at \((x, y)\) equal to

\[ \Delta \hat{f}_\phi(x, y) = \frac{\epsilon \phi(x \cos(k_0 \Delta \theta) + y \sin(k_0 \Delta \theta) - t_{j_0})}{N(M+1)} = \frac{\epsilon}{N(M+1)} \phi( \text{dist}( (x, y), l_{t_{j_0}, \omega(k_0 \Delta \theta))} ). \]

The effect of this error at \((x, y)\) depends on the distance from \((x, y)\) to the “bad ray,” \( l_{t_{j_0}, \omega(k_0 \Delta \theta)} \). In light of the form of the function \( \phi \) the error is worst along the ray itself, where it equals

\[ \frac{\epsilon N}{\pi(M+1)}. \]

**Example 12.4.1.** In this example we use the Shepp-Logan phantom with 180 views and 367 rays in each view. The rays are numbered so that ray number 184 passes through the center of the image. Figure 12.21 shows a reconstruction of the Shepp-Logan phantom with errors in several rays. We list rays in the form \((j, \theta)\) where \( j \) is the ray number and \( \theta \) is in degrees. Rays (220, 30), (221, 30), (250, 90), (251, 90) each have a %10 error, rays (190, 180), (191, 180) have a %20 error and rays (110, 135), (111, 135) have a %30 error. We introduce an error in two adjacent rays to get a clearer picture of the artifact.

![A reconstruction of the Shepp-Logan phantom with a few isolated bad rays.](image-url)
12.4.2 A bad ray in each view

A bad ray might result from a momentary surge in the output of the x-ray tube. If, on the other hand, a single detector in the detector array is malfunctioning then the same ray in each view will be in error. Let \( \epsilon_k \) denote the error in \( P(t_{j0}, \omega(k\Delta\theta)) \). In light of the linearity of (12.25), the error at \((x, y)\) is now

\[
\Delta \tilde{f}_\phi(x, y) = \sum_{k=0}^{M} \epsilon_k \phi(x \cos(k\Delta\theta) + y \sin(k\Delta\theta) - t_{j0}) \quad \frac{N(M + 1)}{}
\]  

(12.26)

If \((x, y) = r(\cos\varphi, \sin\varphi)\) in polar coordinates and \(\epsilon = \epsilon_k\) for all \(k\) then

\[
\Delta \tilde{f}_\phi(x, y) = \sum_{k=0}^{M} \epsilon \phi(r \cos(\varphi - k\Delta\theta) - t_{j0}) \quad \frac{N(M + 1)}{}
\]

\[
\approx \frac{\epsilon}{\pi N} \int_{0}^{\pi} \phi(r \cos(\varphi - s) - t_{j0}) ds.
\]

(12.27)

Because the function \(\phi\) is sharply peaked at zero and

\[
\int_{-\infty}^{\infty} \phi(s) ds = 0,
\]

this artifact is worst for points where \(r = t_{j0}\) and \(0 < \varphi < \pi\). At other points the integrand is either uniformly small or the integral exhibits a lot of cancellation. Due to the periodicity, of the integrand, in \(\varphi\) it is clear that the largest error occurs where \(r = t_{j0}\) and \(\varphi = \frac{\pi}{2}\). The reason the error is only large in half the circle is that samples are only collected for \(0 \leq \theta \leq \pi\). If data is collected over the full circle then the result of an error \(\epsilon\) in the \(j_{0}\)th ray is approximately

\[
\Delta \tilde{f}_\phi \approx \frac{\epsilon}{2\pi N} \int_{0}^{2\pi} \phi(r \cos(s) - t_{j0}) ds.
\]

(12.28)

If the data is collected over half the circle then \([4\pi]^{-1}\Delta \tilde{f}_\phi\) is the average error for points on the circle of radius \(r\). Figure 12.22 shows graphs of the average error as a function of \(r\) with \(t_{j0} = 0, .25, .5\), and .75. The integral in (12.28) is difficult to numerically evaluate if \(\phi\) is a linearly interpolated function. Instead we have used the approximate filter function

\[
\phi(t) = \frac{d^2 - t^2}{(d^2 + t^2)^2},
\]

which was introduced in section 6.4. These graphs bear out the prediction that the error is worst where \(r = t_{j0}\), moreover the sharpness of the peak also depends on \(t_{j0}\).
12.4. THE EFFECTS OF MEASUREMENT ERRORS

Figure 12.22: Radial graphs of the average error with systematic bad rays at different affine parameters.

Since all the bad rays have the same affine parameter, $t_{j_0}$, they are all tangent to the circle, centered at $(0, 0)$ of radius $t_{j_0}$. In [73] it is shown that the average error along this circle is given approximately by

$$
\frac{\epsilon \sqrt{N}}{\pi^2 \sqrt{t_{j_0}}} \text{ if } t_{j_0} \gg 0 \text{ and } \frac{\epsilon N}{\pi} \text{ if } t_{j_0} = 0.
$$

Figure 12.23: Reconstructions of the Shepp-Logan phantom with projection data having errors in a pair of rays in every view. Note the semi-circular artifacts.

Example 12.4.2. The set-up is the same as in example 12.4.1. The images in figure 12.23 show reconstructions of the Shepp-Logan phantom with an error in a pair of adjacent rays.
in every view. In figure 12.23(a), rays 200 and 201 have an error that is at most %1 in every view. In figure 12.23(b), rays 90 and 91 have an error that is at most %2 in every view.

Exercises

Exercise 12.4.1. Explain why the error is only large in a semi-circle if samples are collected for $0 \leq \theta \leq \pi$.

Exercise 12.4.2. Prove that $[4\pi]^{-1} \Delta \tilde{f}_\phi(r)$ represents the average error on the circle of radius $r$ if data is collected for $0 \leq \theta \leq \pi$.

12.4.3 A bad view

In a third generation machine a single bad detector would result in the situation analyzed in the previous section: the measurement of the same ray would be erroneous in every view. This is because a view, for a third generation machine, is determined by the source position. In a fourth generation scanner a mis-calibrated detector could instead result in every ray from a single view being in error. This is because a view is determined, in a fourth generation machine, by a detector.

We now analyze the effect on the reconstruction of having an error in every measurement from a single view. As before we assume the image is reconstructed using the parallel beam algorithm. Suppose that $\epsilon_j$ is the error in the measurement $P(t_j, \omega(k_0\Delta\theta))$, then the reconstruction error at $(x, y)$ is

$$\Delta \tilde{f}_\phi(x, y) = \frac{1}{N(M + 1)} \sum_{j=1}^{N} \epsilon_j \phi((x, y), \omega(k_0\Delta\theta)) - t_j).$$

If the error $\epsilon_j = \epsilon$ for all rays in the $k_0^{th}$ view then the error can be approximated by an integral

$$\Delta \tilde{f}_\phi(x, y) \approx \frac{\epsilon}{2(M + 1)} \int_{-1}^{1} \phi((x, y), \omega(k_0\Delta\theta)) - t) \, dt. \quad (12.29)$$

As before, the facts that $\phi$ is even and has total integral zero make this artifact most severe at the points $\langle(x, y), \omega(k_0\Delta\theta)\rangle = \pm(1 - d)$ and least severe along the line

$$\langle(x, y), \omega(k_0\Delta\theta)\rangle = 0.$$

This is the "central ray" of the $k_0^{th}$ view. From the properties of $\phi$ we conclude that the worst error

$$\max(\Delta \tilde{f}_\phi) \propto \frac{\epsilon d}{8\pi(M + 1)} \phi(0) = \frac{\epsilon N}{2\pi(M + 1)}.$$

On the other hand for points near to $(0, 0)$ the error is approximately

$$\frac{\epsilon}{2(M + 1)} \int_{-1+\delta}^{1+\delta} \phi(t) \, dt,$$
where $\delta$ is the distance between $(x, y)$ and the central ray of the bad view. The integral from $(-1 + \delta)$ to $(1 - \delta)$ is approximated by $\frac{2}{\pi(1-\delta)}$ and the integral from $(1 - \delta)$ to $(1 + \delta)$ is $O(\delta)$, hence

$$\Delta \tilde{f}(x, y) \approx \frac{\epsilon}{\pi(M + 1)} \frac{1}{1 - \delta}.$$  

**Example 12.4.3.** The set-up is again as in example 12.4.1. The values of the projections in view 0 have all been divided by 2. Indeed the estimate above is overly cautious and the change in the image produced by such errors in a single view is hardly discernible. Figure 12.24(a) shows the Shepp-Logan phantom with these systematic errors in view 0. Figure 12.24(b) is the difference of the image in (a) and the reconstruction obtained without errors in the data. It shows that the error is most pronounced where, as predicted, $\langle \hat{x}, y, \omega \rangle \approx \pm 1$. The gray scale in (b) has been compressed in order to make these vertical lines visible. This example indicates that even a large, smoothly varying error has very little effect on the reconstructed image. This is consistent with the smoothness principle.

![Image](a) The reconstruction of the Shepp-Logan phantom with every projection in view 0 divided by 2

![Image](b) The difference of the image in (a) and the reconstruction without systematic errors in the data.

Figure 12.24: The error produced in the reconstruction with a single bad view.

**Remark 12.4.1.** Many other artifacts have been analyzed in [73] and [36]. We have selected examples that have a fairly simple mathematical structure and illustrate the usage of the tools developed in the earlier chapters. In large part due to their successful analyses, these artifacts are largely absent from modern CT-images.

**Exercises**

**Exercise 12.4.3.** Justify the approximation as an integral in (12.29).

**Exercise 12.4.4.** Describe the qualitative properties of the measurement error which would result from a momentary surge in the x-ray source in a fourth generation CT-scanner.
Exercise 12.4.5. Suppose that every measurement is off by $\epsilon$. Show that the reconstructed image has an error

$$\Delta f_\phi(x, y) \approx \frac{\epsilon}{\pi \sqrt{1 - x^2 - y^2}}.$$ 

12.5 Beam hardening

We close our discussion of imaging artifacts with a very short discussion of beam hardening. Because it is non-linear, beam hardening is qualitatively quite different from the foregoing phenomena. It is, instead rather similar to the partial volume effect. Beam hardening is caused by the fact that the x-ray beam is not monochromatic and the attenuation coefficient, depends, in a non-trivial way, on the energy of the incident beam. Recall that an actual measurement is the ratio $I_i/I_o$, where $I_i$ is the total energy in the incident beam and $I_o$ is the total energy in the output. The energy content of the x-ray beam is described by its spectral function, $S(\mathcal{E})$; it satisfies

$$I_i = \int_{0}^{\infty} S(\mathcal{E}) d\mathcal{E}.$$ 

If a (thin) x-ray beam is directed along the line $l_{t, \omega}$, then the measured output is

$$I_{o, (t, \omega)} = \int_{0}^{\infty} S(\mathcal{E}) \exp \left[ -\int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega; \mathcal{E}) ds \right] d\mathcal{E}.$$ 

Here $f(x, y; \mathcal{E})$ is the attenuation coefficient, with its dependence on the energy explicitly noted. A typical spectral function is shown in figure 3.8. Due to this non-linear distortion, the raw measurements are not the Radon transform of $f$; in the imaging literature it is often said that such measurements are inconsistent. Applying the Radon inversion formula to such data leads to streaking in the reconstructed image, see figure 12.25.

Figure 12.25: Streaks caused by beam hardening.
Suppose that $D$ is a bounded object whose attenuation coefficient, $f(\mathcal{E})$ only depends on the energy. Even in this case, the function,

$$\log \left[ \frac{I_i}{I_{o,(t,\omega)}} \right]$$

is not a linear function of length of the intersection of the line $l_{i,\omega}$ with $D$. If $T$ denotes the length of this line segment then

$$\log \left[ \frac{I_i}{I_{o,(t,\omega)}} \right] = H_f(T) = \log \left[ \frac{\int S(\mathcal{E}) e^{-T f(\mathcal{E})} d\mathcal{E}}{\int S(\mathcal{E}) d\mathcal{E}} \right].$$

(12.30)

Because $S(\mathcal{E})$ and $f(\mathcal{E})$ are non-negative functions it is immediate that $H_f(T)$ is a strictly, monotonely decreasing function. This implies that the inverse function, $H_f^{-1}$ is well defined. Thus by measuring or computing $H_f(T)$, for $T$ in the relevant range, its inverse function can be tabulated. The attenuation coefficient of water, $f_w(\mathcal{E})$ as well as $H_w(T)$, for a typical spectral function, are shown in figure 12.26(a-b).  

![Figure 12.26: Beam hardening through water.](image)

Using $H_f^{-1}$ the Radon transform of $\chi_D$ can be determined from x-ray attenuation measurements

$$R\chi_D(t, \omega) = H_f^{-1} \left( \log \left[ \frac{I_i}{I_{o,(t,\omega)}} \right] \right).$$

(12.31)

The data for figure 12.26(a) is from the N.I.S.T. website http://physics.nist.gov/PhysRefData/Xray-MassCoeff/cover.html.
The region $D$ could now be reconstructed using the methods described above for approximately inverting the Radon transform.

The attenuation coefficients of the soft tissues in the human body are close to that of water and their dependence on the energy is similar. If $f_w(E)$ is the attenuation coefficient of water then this amounts to the statement that the ratio

$$\rho(x,y) = \frac{f(x,y;E)}{f_w(E)}$$

is nearly independent of the energy. Let $H_w$ denote the function defined in (12.30) with $f = f_w$. For slices which contain little or no bone the function $H_w^{-1}$ can be used as in (12.31) to correct for beam hardening. This substantially reduces the inconsistencies in the measurements and allows the usage of the Radon formalism to reconstruct $\rho$.

The measurement is re-expressed in terms of $\rho$ by

$$I_{o_i}(t,\omega) = \int_0^\infty S(E) \exp \left[ -f_w(E) \int_{-\infty}^{\infty} \rho(s\omega + t\omega)ds \right] dE.$$  

Applying $H_w^{-1}$ to these measurements gives the Radon transform of $\rho$,

$$H_w^{-1} \left( \log \left[ \frac{I_{o_i}}{I_{o_i}(t,\omega)} \right] \right) = \int_{-\infty}^{\infty} \rho(s\omega + t\omega)ds.$$  

The function $\rho$ is a non-negative function which reflects the internal structure of the slice in much the same way as a mono-energetic, attenuation coefficient.

(a) Dark streaks in a phantom with dense objects.  
(b) Dark streaks in an image of the head produced by bone.

Figure 12.27: Due to beam hardening, dense objects produce dark streaks.

Having materials of very different densities in a slice leads to a much more difficult beam hardening problem; one which is, as of this writing, not completely solved. In x-ray CT
this is principally the result of bones intersecting the slice. It causes dark streak artifacts as seen in figure 12.27. The analysis of this problem is beyond the scope of this text. In [38] an effective algorithm is presented to substantially remove these artifacts. Another method, requiring two sets of measurements with x-ray beams having different spectral functions, is described in [2]. The discussion in this section is adapted from [38].

Exercises

Exercise 12.5.1. Prove that \( H_f(T) \) is a strictly monotone decreasing function.

Exercise 12.5.2. Find a Taylor series expansion for \( H_f^{-1} \).

12.6 Conclusion

The introduction of the mathematical phantom and the mathematical analysis of errors in algorithms were essential steps in the removal of imaging artifacts. Figure 12.28(a) shows a CT-image of a slice of the head from a 1970s era machine. Note the enormous improvement in the 1990s era image of the same section, shown in figure 12.28(b).

(a) 1970s era brain section.  
(b) 1990s era brain section.

Figure 12.28: Mathematical analysis has led to enormous improvements in CT-images.