THE MARKED LENGTH-SPECTRUM OF A SURFACE OF NONPOSITIVE CURVATURE

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Abstract. We generalize work of J.P. Otal and C. Croke on the marked length spectrum of surfaces to the case where one of the metrics is of nonpositive curvature and the other one has no conjugate points.

If $M$ is a manifold and $g_1, g_2$ are two Riemannian metrics, we say that they have the same marked length spectrum if in each homotopy class of closed curves in $M$ the infimum of $g_1$-lengths of curves and the infimum of $g_2$-lengths of curves are the same. The marked length spectrum problem in general is to show that two metrics with the same marked length spectrum are isometric. Of course, this cannot hold for arbitrary metrics (for example if $M$ is simply connected). This problem was stated as a conjecture in [BK] in the case where $M$ is a closed surface and $g_1$ and $g_2$ are of negative curvature. This conjecture was solved by J.P. Otal [To] and independently by C. Croke [Cr]—see also [Fa]. Previous work on the problem was done by Guillemin and Kazhdan [GK].

In this work, using Otal's approach, we improve some of these results by proving the following theorem:

**Theorem A.** Let $M$ be a closed surface and let $g_1, g_2$ be Riemannian metrics on $M$, with $g_1$ of nonpositive curvature and $g_2$ without conjugate points. If $g_1$ and $g_2$ have the same marked length-spectrum then they are isometric by an isometry homotopic to the identity.

We will also prove the following fact, which reduces the length spectrum and curvature condition to the assumption that the Morse correspondence preserves angles—see §1 for the definition of the Morse correspondence.

**Theorem B.** Let $M$ be a closed surface of genus $\geq 2$, and let $g_1, g_2$ be Riemannian metrics without conjugate points on $M$. If $g_1$ and $g_2$ have the same marked length-spectrum and the Morse correspondence preserves angles then they are isometric by an isometry homotopic to the identity.
Finally, we obtain a third result of a more dynamical nature. This is a generalization of a question raised in OF, 6.3 page 70, see also [Cr] and CF where this question is solved.

**Theorem C.** Let \((M_1, g_1)\) and \((M_2, g_2)\) be Riemannian closed surfaces of genus \(\geq 2\) without conjugate points. If one of the two metrics has nonpositive curvature, then any time preserving semi-conjugacy from the geodesic flow of \((M_1, g_1)\) to the geodesic flow of \((M_2, g_2)\) comes from a Riemannian submersion composed with a shift by some fixed time.

We introduce here some definitions and notation. If \(g\) is a Riemannian metric on the surface \(M\) we will denote by \(\kappa_g(m)\) the curvature of \(g\) at a point \(m \in M\). The lift of \(g\) to the universal cover \(\tilde{M}\) of \(M\) will be denoted by \(\tilde{g}\). A \(\tilde{g}\)-strip in \(\tilde{M}\) is a closed subset of \(\tilde{M}\) homeomorphic to \(\mathbb{R} \times [0, 1]\) whose boundary consists of two \(\tilde{g}\)-geodesics which remain at bounded distance from each other. Any two disjoint \(\tilde{g}\)-geodesics which remain at bounded distance from one another and are closed as subsets of \(\tilde{M}\) bound a \(\tilde{g}\)-strip. A \(\tilde{g}\)-strip is flat if the curvature of \(\tilde{g}\) is zero on the strip. If two \(\tilde{g}\)-geodesics \(G\) and \(G'\) intersect at a unique point, we will denote by \(\angle_{\tilde{g}}(G, G') \in [0, \pi]\) the angle at the point of intersection.

1. Background.

We fix a reference Riemannian metric \(g_0\) of (strictly) negative curvature on \(M\). The following theorem is due to Morse [Mr].

**Theorem 1.1. (Morse).** Let \(g\) be a Riemannian metric on \(M\). Let \(\tilde{g}\) and \(\tilde{g}_0\) be the lifts of \(g\) and of the Riemannian metric \(g_0\) of (strictly) negative curvature to the universal cover \(\tilde{M}\) of \(M\). Then there exists a constant \(K > 0\), which depends only on \(g\) and \(g_0\), such that any \(\tilde{g}_0\)-geodesic contains in its \(K\)-neighborhood a \(\tilde{g}\)-minimizing \(\tilde{g}\)-geodesic, and any minimizing \(\tilde{g}\)-geodesic contains a unique \(\tilde{g}_0\)-geodesic in its \(K\)-neighborhood. The map \(\tilde{S}: \tilde{M} \to \tilde{G}_0\) from the space \(\tilde{M}\) of \(\tilde{g}\)-minimizing geodesics to the space \(\tilde{G}_0\) of \(\tilde{g}_0\)-geodesics which sends a minimizing \(\tilde{g}\)-geodesic to the \(\tilde{g}_0\)-geodesic in its \(K\)-neighborhood is continuous and proper. (When \(g\) has no conjugate points then \(\tilde{M}\) is of course the space \(\tilde{G}\) of all \(\tilde{g}\)-geodesics.) Moreover, the map \(\tilde{S}\) is \(\pi_1(M)\) equivariant.

We will call the map \(\tilde{S}\) a Morse map.

**Corollary 1.2.** For every Riemannian metric \(g\) without conjugate points on \(M\), there is a constant \(C\) such that any \(\tilde{g}\) strip in \(\tilde{M}\) has width \(\leq C\).

**Proof.** The Morse map sends all \(\tilde{g}\)-geodesics entirely contained in a given strip to the same \(\tilde{g}_0\)-geodesic.

It will be convenient to introduce the following concept. If \(g_1\) and \(g_2\) are metrics on \(M\), we say that the \(\tilde{g}_1\)-geodesic \(G_1\) and the \(\tilde{g}_2\)-geodesic \(G_2\), both in \(\tilde{M}\), are Morse
correspondent and we will write $G_1 \sim G_2$, if they remain at bounded distance, i.e. if we have:

$$\sup_{m \in G_1} d(m, G_2) < \infty \quad \text{and} \quad \sup_{m \in G_2} d(m, G_1) < \infty,$$

where $d$ is a metric on $\tilde{M}$ coming from a Riemannian metric on $M$. Since $M$ is compact, the condition does not depend on the choice of $d$. It is not assumed that the Riemannian metrics $g_1$ and $g_2$ are distinct. As before $g_0$ will be a fixed reference Riemannian metric of (strictly) negative curvature on $M$. Suppose $g_1$ and $g_2$ are Riemannian metrics without conjugate points on $M$. Call $\tilde{\mathcal{S}}_1 : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_0$ (resp. $\tilde{\mathcal{S}}_2 : \tilde{\mathcal{G}}_2 \to \tilde{\mathcal{G}}_0$) the Morse map from the set of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) onto the set of $\tilde{g}_0$-geodesics. If $G_1$ is a $\tilde{g}_1$-geodesic and $G_2$ is a $\tilde{g}_2$-geodesic, then $G_1 \sim G_2$ if and only if $\tilde{\mathcal{S}}_1(G_1) = \tilde{\mathcal{S}}_2(G_2)$. The following two statements are easy consequences of the work of Leon Green [Gr, corollary 3.2 page 536 and theorem 4.1 page 559]:

**Proposition 1.3.** Let $g$ be a Riemannian metric without conjugate points on $M$. Let $G$ and $G'$ be two $\tilde{g}$-geodesics (in $\tilde{M}$). If $G \sim G'$ then either $G = G'$ or $G$ and $G'$ do not intersect; moreover, in the latter case they bound a $\tilde{g}$-strip, and through each point $y$ of this $\tilde{g}$-strip there passes a unique $\tilde{g}$-geodesic $G_y$ such that $G \sim G_y$. If $g$ is of nonpositive curvature then all $\tilde{g}$-strips are flat.

**Corollary 1.4.** Let $g_1$ and $g_2$ be Riemannian metrics without conjugate points on $M$. Let $G_1$ and $G_1'$ (resp. $G_2$ and $G_2'$) be two $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) in $\tilde{M}$. If $G_1 \sim G_2$ and $G_1' \sim G_2'$ then $G_1$ and $G_1'$ intersect transversally if and only if $G_2$ and $G_2'$ intersect transversally.

We will now show how to adapt Otal’s arguments [To] to prove the following lemma:

**Lemma 1.5.** Let $g_1$ and $g_2$ be two Riemannian metrics without conjugate points on $M$. Suppose that $g_2$ is of nonpositive curvature and that $g_1$ and $g_2$ have the same length spectrum. Then:

For every pair $(G_1, G_1')$ of $\tilde{g}_1$-geodesics, and every pair $(G_2, G_2')$ of $\tilde{g}_2$-geodesics, if $G_1 \sim G_2$ and $G_1' \sim G_2'$, then $\angle_{\tilde{g}_1}(G_1, G_1') = \angle_{\tilde{g}_2}(G_2, G_2')$.

Consequently the metric $g_1$ also has nonpositive curvature.

**Sketch of proof.** We will use the setting of [Fa] to show how to adapt the arguments of Otal. Let $g_0$ be a metric of (strictly) negative curvature on $M$. Let $\tilde{\mathcal{S}}_1 : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}$ (resp. $\tilde{\mathcal{S}}_2 : \tilde{\mathcal{G}}_2 \to \tilde{\mathcal{G}}$) be the Morse map from the space of $\tilde{g}_1$-geodesics (resp. $\tilde{g}_2$-geodesics) onto the space of $\tilde{g}_0$-geodesics, as described above. As in [Fa], using $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$, we obtain, from the Liouville measures, geodesic currents $\tilde{\lambda}_{g_1}$ and $\tilde{\lambda}_{g_2}$. By [To, théorème 2], we obtain $\lambda_{g_1} = \lambda_{g_2}$. We define for each pair $(G, G')$ of transversally intersecting $\tilde{g}_0$-geodesics the angle $\theta'(G, G')$ as the $\tilde{g}_2$ angle of any pair $(G_2, G_2')$ of $\tilde{g}_2$-geodesics such that $\tilde{\mathcal{S}}_2(G_2) = G$ and $\tilde{\mathcal{S}}_2(G_2') = G'$. The fact that this angle is independent of the choices follows from the flatness of the $\tilde{g}_2$-strips—see lemma 1.3. It is not very
difficult now to adapt Otal’s arguments [To] as in [Fa], to prove the angle condition of the lemma.

This angle condition, taken with the fact that \( g_2 \) has nonpositive curvature, implies that the sum of the angles of any triangle whose sides are \( \tilde{g}_1 \)-geodesics is \( \leq \pi \). It follows from the Gauss-Bonnet theorem that \( g_1 \) also has nonpositive curvature.

2. Proof of theorem A.

Because the sphere is simply connected the genus of \( M \) has to be \( \geq 1 \). A theorem of Hopf says that a metric without conjugate points on a torus or a Klein bottle is flat. Theorem A follows for the torus and the Klein bottle—see [Cr, pages 167–168]. So we assume for the rest of the section that the genus of \( M \) is \( \geq 2 \). Since we want to use the work done in this section to prove other theorems, we will use general arguments as often as possible.

**Definition 2.1.** Suppose \( g_1 \) and \( g_2 \) are Riemannian metrics without conjugate points on \( M \). We define a partial relation \( R \) on \( \tilde{M} \) in the following way: \( m \sim m' \) if every \( \tilde{g}_2 \)-geodesic through \( m' \) is at bounded distance from some \( \tilde{g}_1 \)-geodesic through \( m \).

**Lemma 2.2.** Suppose \( g_1 \) and \( g_2 \) are Riemannian metrics without conjugate points on \( M \). If \( m \sim m' \), then every \( \tilde{g}_1 \)-geodesic through \( m \) is at bounded distance from some \( \tilde{g}_2 \)-geodesic through \( m' \).

**Proof.** Let \( g_0 \) be a metric of (strictly) negative curvature on \( M \). As before, let \( \tilde{S}_1 : \tilde{G}_1 \to \tilde{G} \) (resp. \( \tilde{S}_2 : \tilde{G}_2 \to \tilde{G} \)) be the Morse map from the space of \( \tilde{g}_1 \)-geodesics (resp. \( \tilde{g}_2 \)-geodesics) onto the space of \( \tilde{g}_0 \)-geodesics. We have \( m \sim m' \) if and only if \( \tilde{S}_2 \{ G \in \tilde{G}_2 \mid m' \in G \} \subset \tilde{S}_1 \{ G \in \tilde{G}_1 \mid m \in G \} \). But by proposition 1.3, the map \( \tilde{S}_1 \) (resp. \( \tilde{S}_2 \)) is injective on \( \{ G \in \tilde{G}_1 \mid m \in G \} \) (resp. \( \{ G \in \tilde{G}_2 \mid m' \in G \} \)). Hence we have a natural 1-1 continuous map from the circle of \( \tilde{g}_2 \)-geodesics through \( m' \) to the circle of \( \tilde{g}_1 \)-geodesics through \( m \). Such a map must be a homeomorphism, so we are done.

**Lemma 2.3.** Suppose \( g_1 \) and \( g_2 \) are Riemannian metrics without conjugate points on \( M \), and every \( \tilde{g}_1 \)-strip is flat. If \( m_1 \sim m' \) and \( m_2 \sim m' \), then \( m_1 = m_2 \). If every \( \tilde{g}_2 \)-strip is also flat, then \( R \) is the graph of a bijection \( \varphi \) between the domain \( \tilde{D}_1 \) of \( R \) and its range \( \tilde{D}_2 \). Both \( \tilde{D}_1 \) and \( \tilde{D}_2 \) are invariant under the action of \( \pi_1(M) \), and moreover, the map \( \varphi \) is \( \pi_1(M) \) equivariant.

**Proof.** From Lemma 2.2 and the definition of \( R \), it follows that if \( G \) is a \( \tilde{g}_1 \)-geodesic through \( m_1 \) it is at bounded distance from some \( \tilde{g}_1 \)-geodesic \( G' \) through \( m_2 \). If \( m_1 \neq m_2 \) and \( G \) is not the geodesic through \( m_1 \) and \( m_2 \), then \( G \) and \( G' \) bound a \( \tilde{g}_1 \)-strip which by hypothesis must be flat. This implies that the curvature of \( \tilde{g}_1 \) along every \( \tilde{g}_1 \)-geodesic through \( m_1 \) is 0 (by continuity this is also true for the
Under the hypothesis of Theorem A, let Lemma 2.4.

Let us now suppose that the Riemannian metrics \( g_1 \) and \( g_2 \) without conjugate points on \( M \) have the same length spectrum and one of them has non positive curvature. By 1.5, both of them have nonpositive curvature, and by 1.3 all \( \tilde{g}_1 \) and \( \tilde{g}_2 \) strips are flat and by 2.3 the relation \( \mathcal{R} \) is the graph of a bijection.

Let \( \tilde{U}_i, i = 1, 2 \), be \( \{ m \in \tilde{M} \mid \kappa_{\tilde{g}_i}(m) \neq 0 \} \).

**Lemma 2.4.** Under the hypothesis of Theorem A, let \( G_1 \) and \( G'_1 \) be \( \tilde{g}_1 \)-geodesics which intersect transversally at a point \( m \) which is in \( \tilde{U}_1 \). Suppose that \( G_2 \) and \( G'_2 \) are \( \tilde{g}_2 \)-geodesics with \( G_1 \sim G_2 \) and \( G'_1 \sim G'_2 \). If \( m' \) is the point of intersection of \( G_2 \) and \( G'_2 \) then \( m \mathcal{R} m' \). In particular, the set \( \tilde{U}_1 \) is contained in the domain \( \tilde{D}_1 \) of the map \( \varphi \) given by 2.3.

Moreover, every \( \tilde{g}_1 \)-geodesic which is at bounded distance from a \( \tilde{g}_1 \)-geodesic through \( m \) must also pass through \( m \), and every \( \tilde{g}_2 \)-geodesic which is at bounded distance from a \( \tilde{g}_2 \)-geodesic through \( m' \) must also pass through \( m' \).

**Proof.** If \( H \) is a \( \tilde{g}_1 \)-geodesic which is at bounded distance from some \( \tilde{g}_2 \)-geodesic which passes through \( m' \), the angle condition of Lemma 1.5 shows that \( G_1, G'_1 \) and \( H \) bound a triangle \( T \) whose sum of angles is \( \pi \). But \( g_1 \) is of nonpositive curvature and one of the vertices of the triangle, namely \( m \), satisfies \( \kappa_{\tilde{g}_1}(m) \neq 0 \), so from the Gauss-Bonnet theorem it follows that \( T \) is degenerate and that \( H \) goes through \( m \). This proves that \( m \mathcal{R} m' \).

Since \( \tilde{g}_1 \)-strips are flat, the point \( m \) cannot be contained in a \( \tilde{g}_1 \)-strip. So every \( \tilde{g}_1 \)-geodesic which is at bounded distance from some \( \tilde{g}_1 \)-geodesic through \( m \) must also pass through \( m \).

It remains to prove that every \( \tilde{g}_2 \)-geodesic which is at bounded distance from a \( \tilde{g}_2 \)-geodesic through \( m' \) must also pass through \( m' \). Let \( H'_2 \) and \( H''_2 \) be two \( \tilde{g}_2 \)-geodesics that remain at bounded distance and suppose that \( m' \in H'_2, m' \notin H''_2 \) and \( m'' \in H''_2 \). We know from the first part that both \( H'_2 \) and \( K \), the \( \tilde{g}_2 \)-geodesic through \( m' \) and \( m'' \), are at bounded distance from \( \tilde{g}_1 \)-geodesics that pass through \( m \). It follows that the pair \( H''_2, K \) of transverse \( \tilde{g}_2 \)-geodesics are at bounded distance from \( \tilde{g}_1 \)-geodesics through \( m \). By the first part of the lemma \( m \mathcal{R} m'' \). From Lemma 2.3, we obtain \( m' = m'' \). This is a contradiction. 

**Lemma 2.5.** Under the hypothesis of Theorem A, if \( m, m' \in \tilde{U}_1 \), then \( d_{\tilde{g}_1}(m, m') = d_{\tilde{g}_2}(\varphi(m), \varphi(m')) \), where \( \varphi \) is given by Lemma 2.3. In particular, the map \( \varphi \) induces an isometry between \( \tilde{U}_1 \) and \( \tilde{U}_2 \).

**Proof.** Fix a Riemannian metric \( g_0 \) on \( M \) of (strictly) negative curvature and let \( \tilde{S}_1 : \tilde{G}_1 \to \tilde{G} \) and \( \tilde{S}_2 : \tilde{G}_2 \to \tilde{G} \) the Morse maps described above. It is not difficult to see, using lemma 2.4, that there exists a set \( A \subset \tilde{G} \) such that \( \tilde{S}_1^{-1}(A) \) (resp. \( \tilde{S}_2^{-1}(A) \)) is the subset of \( \tilde{G}_1 \) (resp. \( \tilde{G}_2 \)) consisting of \( \tilde{g}_1 \)-geodesics (resp. \( \tilde{g}_2 \)-geodesics) that intersect the \( \tilde{g}_1 \)-geodesic (resp. \( \tilde{g}_2 \)-geodesic) segment between \( m \) and \( m' \) (resp. \( \varphi(m) \) and \( \varphi(m') \)). Using the fact that the Liouville currents obtained from \( g_1 \) and
Moreover, the extension \(\bar{\psi}\) of the map will be an isometry at points of \(\bar{\tilde{\phi}}\).

Proof. As above \(M\) is a Riemannian isometry of points on \(\tilde{M}\). It follows that \(\bar{\psi}\) fields and the derivative of the exponential map—see \([Kl, lemma 5.4.3 page 102]\)—it is isometric by an isometry homotopic to the identity of \(M\).

Lemma 2.6. Let \(g_1\) and \(g_2\) be Riemannian metrics without conjugate points on \(M\), for which \(\tilde{g}_1\) and \(\tilde{g}_2\) strips are flat. Suppose that the map \(\varphi\) induces a bijection between \(\tilde{U}_1 = \{m \in \tilde{M} \mid \kappa_{g_1}(m) \neq 0\}\) and \(\tilde{U}_2 = \{m \in \tilde{M} \mid \kappa_{g_2}(m) \neq 0\}\), and that for every \(m, m' \in \tilde{U}_1\), we have \(d_{\tilde{g}_1}(m, m') = d_{\tilde{g}_2}(\varphi(m), \varphi(m'))\). Then \(\varphi\) extends to a Riemannian isometry of \((\tilde{M}, \tilde{g}_1)\) onto \((\tilde{M}, \tilde{g}_2)\) which is equivariant under the action of \(\pi_1\). Hence the Riemannian metrics \(g_1\) and \(g_2\) are isometric by an isometry homotopic to the identity of \(M\).

Proof. As above \(\varphi\) induces a Riemannian isometry. If \(p \in \tilde{U}_1\), call \(T_p\bar{\varphi}\) the derivative of \(\varphi\) at \(p\). One can check that the map \(\bar{\varphi} = \exp_{\bar{\varphi}p} T_p\varphi(\exp_{\bar{\psi}p}^{-1})\) extends \(\varphi\) to \(\tilde{M}\). Moreover, the extension \(\bar{\varphi}\) preserves curvature, since along any geodesic through \(p\) the map will be an isometry at points of \(\tilde{U}_1\), namely \(\varphi\), and will take points of zero curvature to points of zero curvature. From the well-known relation between Jacobi fields and the derivative of the exponential map—see \([Kl, lemma 5.4.3 page 102]\)—it follows that \(\bar{\varphi} : (\tilde{M}, \tilde{g}_1) \to (\tilde{M}, \tilde{g}_2)\) is an isometry. The fact that \(\bar{\varphi}\) is equivariant under \(\pi_1(M)\) follows from the fact that \(\varphi\) is invariant under the same action. \(\square\)

The proof of theorem A follows from the above lemmas.

3. Proof of theorem B.

In this section we assume that \(g_1\) and \(g_2\) are Riemannian metrics without conjugate points on \(M\), and that the angle hypothesis of Theorem B is satisfied, i.e.:

For every pair \((G_1, G'_1)\) of \(\tilde{g}_1\)-geodesics, and every pair \((G_2, G'_2)\) of \(\tilde{g}_2\)-geodesics,

\[
G_1 \sim G_2 \quad \& \quad G'_1 \sim G'_2 \quad \Rightarrow \quad \angle_{\tilde{g}_1}(G_1, G'_1) = \angle_{\tilde{g}_2}(G_2, G'_2).
\]

As before let \(\tilde{U}_i = \{m \in \tilde{M} \mid \kappa_{\tilde{g}_i}(m) \neq 0\}, i = 1, 2\).

We first prove three more lemmas.

Lemma 3.1. All strips for \(\tilde{g}_1\) and \(\tilde{g}_2\) are flat. Consequently, no point of \(\tilde{U}_1\) (resp. \(\tilde{U}_2\)) is contained in a \(\tilde{g}_1\) (resp. \(\tilde{g}_2\)) strip.

Proof. We will show the result for \(\tilde{g}_1\). A consequence of the hypothesis of theorem 3.1, is that if the \(\tilde{g}_1\)-geodesics \(G, G'\) remain at bounded distance then any other \(\tilde{g}_1\)-geodesic cuts them at the same angle. By the result of Leon Green, Proposition 1.3, any strip bounded by two \(\tilde{g}_1\)-geodesics can be foliated by infinite \(\tilde{g}_1\)-geodesics. It is easy to deduce that any point inside the strip is contained in arbitrarily small geodesic triangles whose sum of interior angles is \(\pi\). It follows from the Gauss-Bonnet theorem that the strip is flat. \(\square\)
Lemma 3.2. Let \( \tilde{U}_1 = \{ G_1 \in \tilde{G}_1 \mid G_1 \cap \tilde{U}_1 \neq \emptyset \} \) and \( \tilde{U}_2 = \{ G_2 \in \tilde{G}_2 \mid \exists G_1 \in \tilde{U}_1, G_1 \sim G_2 \} \). The formula \( G_2 \sim \tilde{S} G_2 \) defines a continuous surjective map \( \tilde{S} : \tilde{U}_2 \to \tilde{U}_1 \). Of course \( \tilde{U}_1 \) is open; moreover, the set \( \tilde{U}_2 \) is also open.

Proof. By Lemma 3.1, if \( G, G' \) are \( \tilde{g}_1 \)-geodesics with \( G \in \tilde{U}_1 \) and \( G \sim G' \) then \( G = G' \). As above, let \( \tilde{S}_1 : \tilde{G}_1 \to \tilde{G} \) and \( \tilde{S}_2 : \tilde{G}_2 \to \tilde{G} \) the Morse maps obtained in 1.1. From the observation just made, \( \tilde{S}_1 \) induces a bijection from \( \tilde{U}_1 \to \tilde{S}_1(\tilde{U}_1) \) and \( \tilde{S}_1^{-1}\tilde{S}_1(\tilde{U}_1) = \tilde{U}_1 \). It is not difficult, using the fact that \( \tilde{S}_1 \) is continuous and proper, to conclude that \( \tilde{S}_1(\tilde{U}_1) \) is open and that \( \tilde{S}_1 \) restricts to a homeomorphism from \( \tilde{U}_1 \) onto \( \tilde{S}_1(\tilde{U}_1) \). The lemma follows since \( \tilde{U}_2 = \tilde{S}_2^{-1}(\tilde{S}_1(\tilde{U}_1)) \) and \( \tilde{S} = \tilde{S}_1^{-1}\tilde{S}_2 \).

Lemma 3.3. Suppose that the \( \tilde{g}_2 \)-geodesics \( G_2, G'_2 \) are in \( \tilde{U}_2 \) and that \( \tilde{S}G_2 \) and \( \tilde{S}G'_2 \) intersect transversally at a point \( m \) which is in \( \tilde{U}_1 \). If \( m' \) is the point of intersection of \( G_2 \) and \( G'_2 \) then \( mRm' \). Moreover, every \( \tilde{g}_1 \)-geodesic which is at bounded distance from some \( \tilde{g}_1 \)-geodesic through \( m \) must also pass through \( m \), and every \( \tilde{g}_2 \)-geodesic which is at bounded distance from some \( \tilde{g}_2 \)-geodesic through \( m' \) must also pass through \( m' \).

Proof. Let \( G_2 \) and \( G'_2 \) be \( \tilde{g}_2 \)-geodesics passing through \( m' \) such that \( G_2 \sim G_1 \) and \( G'_2 \sim G'_1 \) where \( G_1 \) and \( G'_1 \) are \( \tilde{g}_1 \)-geodesics passing trough \( m \). For \( \theta \in [0, \pi] \), let \( G^\theta_2 \) (resp. \( G'^\theta_2 \)) be the \( \tilde{g}_2 \)-geodesic through \( m' \) making an angle \( \theta \) with \( G_2 \) (resp. \( G'_2 \)). Let \( T \) be the set of \( \theta \in [0, \pi] \) such that there exists \( \tilde{g}_1 \)-geodesics \( G_1^\theta \) and \( G'_1^\theta \) through \( m \) with \( G^\theta_2 \sim G_1^\theta \) and \( G'^\theta_2 \sim G'_1^\theta \).

Since two geodesics that stay at a bounded distance must stay within a constant distance depending only on \( g_0, g_1 \) and \( g_2 \) we see that \( T \) must be closed.

Since \( T \) is non-empty we need only show that \( T \) is open. Let \( \psi \in T \). Since \( G^\psi_2 \) and \( G'^\psi_2 \) are in \( \tilde{U}_2 \) and \( \tilde{U}_2 \) is open, if \( \theta \) is close enough to \( \psi \), the \( \tilde{g}_2 \)-geodesics \( G^\theta_2 \) and \( G'^\theta_2 \) are also in \( \tilde{U}_2 \) and hence there exists unique geodesics \( G_1^\theta \) and \( G'_1^\theta \) with \( G^\theta_2 \sim G_1^\theta \) and \( G'^\theta_2 \sim G'_1^\theta \).

Let \( \epsilon > 0 \) be so small that \( B(m, \epsilon) \), the \( \epsilon \)-ball for \( \tilde{g}_1 \) about \( m \), is convex and has non-zero curvature for \( \tilde{g}_1 \) at every point. By transversality and continuity of the map \( \tilde{S} : \tilde{U}_2 \to \tilde{U}_1 \) we see that for all \( \theta \) sufficiently close to \( \psi \) the intersection points of \( G_1^\theta \) and \( G'_1^\theta \), \( G_1^\psi \) and \( G'_1^\psi \), \( G_1^\theta \) and \( G'_1^\theta \), and \( G_1^\theta \) and \( G'_1^\psi \) all lie in \( B(m, \epsilon) \).

We consider two cases. First assume \( G^\theta_2 \) intersects \( G_1^\psi \) inside \( B(m, \epsilon) \) (or similarly that \( G'_1^\theta \) intersects \( G'_1^\psi \) inside \( B(m, \epsilon) \)). Then the geodesic triangles \( G_1^\theta, G_1^\psi, G'_1^\psi \) and \( G'_1^\psi, G'_1^\theta, G'_1^\theta \) both lie inside \( B(m, \epsilon) \). By preservation of angles both have interior angles that sum to \( \pi \), but since the curvature is never zero in these triangles, the Gauss-Bonnet theorem forces them to be degenerate triangles which forces all these geodesics to pass through the common point \( m \), so \( mRm' \).

If on the other hand, both the intersections of \( G_1^\theta \) with \( G_1^\psi \) and \( G'_1^\psi \) with \( G'_1^\theta \) occur outside \( B(m, \epsilon) \) we see that \( G_1^\theta, G_1^\psi, G'_1^\theta, G'_1^\psi \) form a quadrilateral inside \( B(m, \epsilon) \). Again the Gauss-Bonnet theorem forces this quadrilateral to be degenerate and all geodesics pass through \( m \), and again \( mRm' \).

Again by flatness of strips, every \( \tilde{g}_1 \)-geodesic at a bounded distance from some \( \tilde{g}_1 \)-geodesic through \( m \) passes through \( m \). Let \( G'_2 \) and \( G''_2 \) be \( \tilde{g}_2 \)-geodesics with
Now Lemma 2.3 yields $m \sim m' \in G_2', m' \notin G_2''$. Let $G_1$ be the $\tilde{g}_1$-geodesic through $m$ with $G_1 \sim G_2'$, hence $G_1 \sim G_2'$. Pick $m'' \in G_2''$ and let $H_2$ be the $\tilde{g}_2$-geodesic through $m'$ and $m''$. We know that there is a $\tilde{g}_1$-geodesic through $m$ such that $H_1 \sim H_2$ since $mRm'$. On the other hand, since $m'' = H \cap G_2''$ the first part of the lemma yields $mRm''$. Now Lemma 2.3 yields $m' = m''$. 

**Corollary 3.4.** Under the assumptions of this section $\tilde{U}_1$ is contained in the domain $\tilde{D}_1$ of the relation $\mathcal{R}$ and $\tilde{U}_2$ is contained in is range $\tilde{D}_2$.

**Proof.** This is immediate from Lemma 3.3. 

**Proof of theorem B.** We now proceed as in the proof of theorem A, using lemma 2.6.

4. **Proof of theorem C.**

Part of the arguments are already in [CF] and [Cr]. It is easy to see, by taking an orientable cover, that one can reduce the proof to the case where $M_1$ is orientable. We will assume that this is the case in the sequel.

**Lemma 4.1.** Suppose $S(M)$ is the unit tangent bundle of the closed surface $M$ of genus $\geq 2$. If a subgroup of $\pi_1(M)$ has non trivial center then it is isomorphic to $\mathbb{Z}$. Call $p_M : S(M) \to M$ the canonical projection. The center of $\pi_1(S(M))$ is contained the kernel of the induced map $p_{M*} : \pi_1(S(M)) \to \pi_1(M)$. Moreover, if $M$ is orientable the kernel of $p_{M*}$ is precisely the center of $\pi_1(S(M))$.

**Proof.** This is well-known and can be proven using elementary hyperbolic geometry.

**Lemma 4.2.** Suppose $M_1$ and $M_2$ are closed surfaces of genus $\geq 2$ endowed respectively with Riemannian metrics $g_1$ and $g_2$. Suppose that $h : S(M_1) \to S(M_2)$ is a time preserving semi-conjugacy between the geodesic flow $g_1^t$ of $g_1$ and the geodesic $g_2^t$ of $g_2$. If $g_2$ has no conjugate points, then $h$ maps the center of $\pi_1(S(M_1))$ in the center of $\pi_1(S(M_2))$, hence it induces a map $h_\# : \pi_1(M_1) \to \pi_1(M_2)$. The map $h_\#$ is injective.

**Proof.** Let us look at the composition $\theta : S(M_1) \to S(M_2) \to M_2$, where the first arrow is $h$ and the second one is $p_{M_2}$. We want to show that $\theta_* : \pi_1(S(M_1)) \to \pi_1(M_2)$ sends the center of $\pi_1(S(M_1))$ to 0. Suppose this is not the case, then by 4.1 the image $G$ of $\theta_*$ is a cyclic subgroup of $\pi_1(M_2)$ which is isomorphic to $\mathbb{Z}$. Let us call $P : C \to M_2$ the covering of $M_2$ such that $P_*(\pi_1(C)) = G$. It is easy to see that $h$ can be written as a composition $S(P)\tilde{h}$ where $\tilde{h} : S(M_1) \to S(C)$ and $S(P) : S(C) \to S(M_2)$ is the tangent map obtained from $P$. If we lift the metric $g_2$
to a metric $\tilde{g}_2$ on $C$ via $P$, we obtain that $\tilde{h}$ is a time preserving semi-conjugacy between flows. Using a little bit of the theory described in §1 and the fact that $C$ is a cylinder or an open Möbius band without conjugate points, it is not difficult to realize that $\tilde{h}$ sends each $g_1$-geodesic to a $\tilde{g}_2$-geodesic that remain in the strip associated to a non-trivial closed $\tilde{g}_2$-geodesic $G$ of minimum length in $C$. If $H$ is a $g_2$-geodesic transversal to $G$, using the fact that $g_2$ has no conjugate points, all $\tilde{g}_2$-geodesics that remain in the strip of $G$ are also transversal to $H$. By the connectedness of $S(M)$, we conclude that $\tilde{h}$ sends each oriented $g_1$-geodesic to a $\tilde{g}_2$-geodesic that always raps around $C$ in the same sense. This is impossible, because a closed oriented $g_1$-geodesic and its opposite are in opposite homotopy classes of closed curves in $S(M)$.

To show that $h_\#$ is injective, let us start with $\gamma$ in $\pi_1(M_1)$, we can find a closed $g_1$-geodesic $G_1$ in the free homotopy class of $\gamma$. Since $h$ is a semi-conjugacy the image $h(G_1) = \bar{\gamma}$ is a closed $g_2$-geodesic. Since $g_2$ has no conjugate points it cannot be homotopic to $0$. 

Suppose $M_1$ and $M_2$ are closed surfaces of genus $\geq 2$ endowed respectively with Riemannian metrics $g_1$ and $g_2$. We assume that $g_2$ has no conjugate points. Since by 4.2, the map $h_\#$ is injective and $M_1$ and $M_2$ are closed surfaces, the subgroup $h_\#(\pi_1(M_1))$ has finite index in $\pi_1(M_2)$ (if not then by covering theory $\pi_1(M_1)$ would be the fundamental group of a connected non-compact surface, but such a group is free and the fundamental group of a closed surface is never free). Hence it is easy to reduce to the case where $h_\#$ is an isomorphism. Since all automorphisms of the fundamental group of a surface can be realized by diffeomorphisms, we can find a diffeomorphism $f : M_2 \to M_1$ such that the induced map $f_* : \pi_1$ is $h_\#^{-1}$. If we use the diffeomorphism $f$ to transport the metric $g_2$ to a metric $\hat{g}_2$ on $M_1$, it is not difficult to see that $\hat{g}_2$ has no conjugate points and that $g_1$ and $\hat{g}_2$ have the same marked length-spectrum. If both $g_1$ and $g_2$ are without conjugate points and one of them of nonpositive curvature, then we can apply theorem A to $g_1$ and $\hat{g}_2$, so if we compose $f$ with an isometry homotopic to the identity, we see that the proof of theorem C is reduced to:

**Lemma 4.3.** Let $g$ be a Riemannian metric with nonpositive curvature on a closed surface $M$. If $h : S(M) \to S(M)$ is a self semiconjugacy of the geodesic flow of $g$ such that $h_\# : \pi_1(M) \to \pi_1(M)$ is the identity then $h = g^t_0$ for some fixed time $t_0$.

**Proof.** It is not difficult to see from the hypothesis on $h_\#$ that we can lift $h$ to a map $\tilde{h} : S(\tilde{M}) \to S(\tilde{M})$ homotopic to the identity by a bounded homotopy, where $\tilde{M}$ is the universal cover of $M$. It follows that for any geodesic $G$ of the lift $\tilde{g}$ of $g$ to the universal cover $\tilde{M}$ the geodesic $\tilde{h}(G)$ is bounded distance from $G$. By Proposition 1.3, the geodesics $G$ and $\tilde{h}(G)$ either coincide or bound a flat strip. Suppose that $G$ is the lift to $\tilde{M}$ of a geodesic dense in $S(M)$; then the second case cannot happen because $G$ has to go through points of negative curvature. In fact, the geodesic $G$ and $\tilde{h}(G)$ have to coincide as oriented geodesics since $\tilde{h}$ preserves time and is homotopic to the identity by a bounded homotopy. The lemma follows easily using the denseness of the image of $G$ in $M$ and the fact that $h$ preserves
Remark 4.4. Suppose $M_1$ and $M_2$ are closed surfaces of genus $\geq 2$ endowed respectively with Riemannian metrics $g_1$ and $g_2$. We assume that $g_2$ has nonpositive curvature. If there exists a time preserving conjugacy (not necessarily $C^1$) between the geodesic flows of $g_1$ and $g_2$, then the arguments in [Cr, lemma 3.2] show that $g_1$ has no conjugate points and we can apply theorem C to obtain [Cr, theorem B] without the assumption that the conjugacy is $C^1$.

References


