Spaces with nonpositive curvature and their ideal boundaries

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Abstract

We construct a pair of finite piecewise Euclidean 2-complexes with nonpositive curvature which are homeomorphic but whose universal covers have nonhomeomorphic ideal boundaries, settling a question from [8].

1.1 Introduction

The ideal boundary of a locally compact Hadamard space $X$ is a compact metrizable space on which the isometry group of $X$ acts by homeomorphisms. Even though the ideal boundary is a well known construct with many applications in the literature (see for example [10, 4, 2]), the action of the isometry group on the boundary has not been studied closely except in the case of symmetric spaces, Gromov hyperbolic spaces, Euclidean buildings, and a handful of other cases. In the Gromov hyperbolic case the boundary behaves nicely with respect to quasi-isometries: any quasi-isometry $f : X_1 \to X_2$ between Gromov hyperbolic Hadamard spaces induces a boundary homeomorphism $\partial_\infty f : \partial_\infty X_1 \to \partial_\infty X_2$ [7]. This has the consequence that the ideal boundary is “geometry independent”:

If a finitely generated group $G$ acts discretely, cocompactly and isometrically on two Gromov hyperbolic Hadamard spaces $X_1, X_2$, then there is a $G$-equivariant homeomorphism $\partial_\infty X_1 \to \partial_\infty X_2$.

In [8, p. 136] Gromov asked whether this fundamental property still holds if the hyperbolicity assumption is dropped. Sergei Buyalo [5] and the authors [6] independently answered Gromov’s question negatively: [5, 6] exhibit a pair of deck group invariant Riemannian metrics on a universal cover which have ideal boundaries homeomorphic to $S^2$, such that the deck group actions on the boundaries are topologically...
inequivalent. Gromov also asked if $\partial_\infty X_1$ must be (non-equivariantly) homeomorphic to $\partial_\infty X_2$ whenever $X_1$ and $X_2$ are Hadamard spaces admitting discrete, cocompact, isometric actions by the same finitely generated group $G$. In this paper we show that even this can fail:

**Theorem 1** There is a pair $\tilde{X}_1, \tilde{X}_2$ of homeomorphic finite 2-complexes with non-positive curvature such that the universal covers $X_1, X_2$ have nonhomeomorphic ideal boundaries.

We remark that if $M_1$ and $M_2$ are closed Riemannian manifolds with nonpositive curvature and $\pi_1(M_1) \simeq \pi_1(M_2)$, then their universal covers will have ideal boundaries homeomorphic to spheres of the same dimension.

Although some basic questions about the boundary have now been answered, a number of related issues are wide open, except in a few special cases. It would be interesting to know exactly which geometric features determine the ideal boundary of a Hadamard space up to (equivariant) homeomorphism. This question has a clean answer (see [6]) in the case of graph manifolds or the 2-complexes considered in this paper. In order to answer the question in any generality, it appears that it will be necessary to develop a kind of “generalized symbolic dynamics” for geodesic flows of nonpositively curved spaces.

### 1.2 Notation and preliminaries

A reference for the facts recalled here is [3]. If $X$ is a Hadamard space, then we denote the ideal boundary of $X$ by $\partial_\infty X$, the geodesic segment joining $x_1, x_2 \in X$ by $\overline{x_1x_2}$, and the geodesic ray leaving $p \in X$ in the asymptote class of $\xi \in \partial_\infty X$ by $\overline{p\xi}$. If $p \in X, \xi_1, \xi_2 \in \partial_\infty X$, then $\angle_p(\xi_1,\xi_2)$ is the angle between the initial velocities of the rays $\overline{p\xi_1}, \overline{p\xi_2}$. $\angle_{\text{Tits}}(\xi_1,\xi_2) := \sup_{p \in X} \angle_p(\xi_1,\xi_2)$ will denote the Tits angle between $\xi_1, \xi_2 \in \partial_\infty X$. If $p \in X$ then $\angle_p(\xi_1,\xi_2) = \angle_{\text{Tits}}(\xi_1,\xi_2)$ iff the rays $\overline{p\xi_1}$ and $\overline{p\xi_2}$ bound a flat sector.

By the Cartan-Hadamard theorem [1, 3], the universal cover of a connected, complete, length space with nonpositive curvature is a Hadamard space with the natural metric. Let $Z$ be a complete, connected space with nonpositive curvature, and let $\pi : \tilde{Z} \to Z$ be the universal cover. If $Y \subset Z$ is a closed, connected, locally convex subset, then the induced length metric on $Y$ has nonpositive curvature, $\pi^{-1}(Y) \subset \tilde{Z}$ is a disjoint union of closed convex components isometric to $\tilde{Y}$, and the induced map $\pi_1(Y) \to \pi_1(Z)$ is a monomorphism.

### 1.3 Leeb complexes

The following piecewise Euclidean 2-complexes were suggested to us by Bernhard Leeb, after a discussion of the graph manifold geometry in [6].

Let $T_0, T_1, T_2$ be flat two-dimensional tori. For $i = 1, 2$, we assume that there are (primitive) closed geodesics $a_i \subset T_0$ and $b_i \subset T_i$ with $\text{length}(a_i) = \text{length}(b_i)$, and we glue $T_i$ to $T_0$ by identifying $a_i$ with $b_i$ isometrically. We assume that $a_1$ and $a_2$ lie in distinct free homotopy classes, and intersect once at an angle $\alpha \in (0, \frac{\pi}{2})$. The resulting
2-complex $\tilde{X}$ is nonpositively curved as a length space because gluing of nonpositively curved spaces along locally convex subsets produces a nonpositively curved space [3]. We refer to $\tilde{X}$ as a **Leeb complex**. For $i = 1, 2$ let $\tilde{Y}_i := T_0 \cup T_i \subset \tilde{X}$. Notice that $\tilde{Y}_1$ and $T_0$ are closed, locally convex subsets of $\tilde{X}$. Therefore the inclusions $\tilde{Y}_i \subset \tilde{X}$ and $T_0 \subset \tilde{X}$ induce monomorphisms of fundamental groups.

### 1.4 The structure of the universal cover

Let $\pi : X \to \tilde{X}$ be the universal covering of $\tilde{X}$. $X$ is a Hadamard space by the Cartan-Hadamard theorem. A **block** is a connected component of $\pi^{-1}(\tilde{Y}_i) \subset X$, and a **wall** is a connected component of $\pi^{-1}(T_0) \subset X$. Let $\mathcal{B}$ and $\mathcal{W}$ denote the collection of blocks and walls in $X$. Each block (resp. wall) is a closed, connected, locally convex subset of $X$ which is intrinsically isometric to the universal cover of $Y_i$ (resp. $T_0$) since the inclusion $\tilde{Y}_i \to \tilde{X}$ (resp. $T_0 \to \tilde{X}$) induces a monomorphism of fundamental groups. Hence each block (resp. wall) is a convex subset of $X$. If $W \in \mathcal{W}$, $B \in \mathcal{B}$, then either $W \cap B = \emptyset$ or $W \cap B = W$ since $W \cap B$ is open and closed in $W$; $W$ is contained in precisely two blocks, one covering $\tilde{Y}_1$ and the other covering $\tilde{Y}_2$. If $B_1, B_2 \in \mathcal{B}$ are distinct blocks and $B_1 \cap B_2 \neq \emptyset$, then (after relabelling if necessary) $B_i$ covers $\tilde{Y}_i$ and so $B_1 \cap B_2$ consists of a (convex) union of walls; therefore $B_1 \cap B_2 = W$ for some $W \in \mathcal{W}$. When $B_1 \cap B_2 \neq \emptyset$ we will say that the blocks $B_1$ and $B_2$ are adjacent.

$\tilde{Y}_i$ is a “flat” $S^1$ bundle over a bouquet of two circles, so the universal cover $Y_i$ of $\tilde{Y}_i$ (and hence each block) is isometric to the metric product of a simplicial tree with $R$. A **singular geodesic of a block** $B$ is the inverse image of a vertex under the projection of $\tilde{B}$ to its tree factor. Note that singular geodesics of adjacent blocks which lie in the common wall intersect at angle $\alpha$.

$\mathcal{B}$ and $\mathcal{W}$ are clearly locally finite collections. The nerve of $\mathcal{B}$ (the simplicial complex recording (multiple) intersections of blocks) is a simplicial tree. To see this note that if $\epsilon > 0$ is sufficiently small and $\mathcal{B}_\epsilon$ is the collection of (open) $\epsilon$-tubular neighborhoods of blocks, then $Nerve(\mathcal{B}_\epsilon)$ is isomorphic to $Nerve(\mathcal{B})$. Using a partition of unity subordinate to this cover of $|Nerve(\mathcal{B}_\epsilon)|$ one gets a continuous map $\phi : X \to |Nerve(\mathcal{B}_\epsilon)|$. Any map $\gamma : S^1 \to |Nerve(\mathcal{B})|$ can be “lifted” to $X$ up to homotopy: there is a $\hat{\gamma} : S^1 \to X$ so that $\pi \circ \hat{\gamma}$ is homotopic to $\gamma$. Since $\pi_1(X)$ is trivial, this implies that $\pi_1(|Nerve(\mathcal{B})|)$ is trivial. In particular, every wall separates $X$.

Our plan is to show that the subspace $\cup_{B \in \mathcal{B}} \partial_\infty B \subset \partial_\infty X$ can be characterized purely topologically\(^3\), and that its topology is different depending on whether $\alpha = \frac{\pi}{2}$ or not. It will then follow that a Leeb complex with $\alpha < \frac{\pi}{2}$ and a Leeb complex with $\alpha = \frac{\pi}{2}$ have universal covers with nonhomeomorphic ideal boundaries.

### 1.5 Itineraries

For each $p \in X \setminus \cup_{W \in \mathcal{W}} W$, $\xi \in \partial_\infty X$, we get a sequence of blocks $B_i$ called the $p$-**itinerary** (simply the **itinerary** if the basepoint $p$ is understood) of $\xi$, as follows.

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\(^3\)At first glance one might think that $\cup_{B \in \mathcal{B}} \partial_\infty B$ is a path component of $\partial_\infty X$, but this turns out not to be the case. It is a “safe” path component, see 1.7.
Let $B_i$ be the $i^{th}$ block that the ray $\overline{p\xi}$ enters; the ray enters a block $B$ if it reaches a point in $B \setminus \bigcup_{W \in W} W$. We will denote the $p$-itinerary of $\overline{p\xi}$ by $\text{Itin}(\overline{p\xi})$ or $\text{Itin}(\xi)$.

**Lemma 2** The itinerary defines a geodesic segment or geodesic ray in the simplicial tree $\text{Nerve}(B)$.

*Proof.* Blocks are convex, so a geodesic cannot revisit any block which it left. The topological frontier of any $B \in B$ is the union of the walls contained in $B$, so a geodesic segment which leaves $B$ must arrive at a wall $W \subset B$, and then enter the block $B' \in B$ adjacent to $B$ along $W$. The collection $B$ is locally finite, so the lemma follows.

Note that $\xi \in \partial_\infty X$ has a finite itinerary if $\xi \in \partial_\infty B$ for some $B \in B$.

### 1.6 Local components of $\partial_\infty X$

Since each block $B$ is isometric to the product of simplicial tree with $R$, $\partial_\infty B$ is homeomorphic to the suspension of a Cantor set. A pole of $B$ is one of the two suspension points in $\partial_\infty B$.

**Lemma 3** If $B_1, B_2 \in B$, then one of the following holds:

1. $\partial_\infty B_1 \cap \partial_\infty B_2 = \emptyset$.
2. $B_1 \cap B_2 = W \in W$ and $\partial_\infty B_1 \cap \partial_\infty B_2 = \partial_\infty W$.
3. There is a $B \in B$ such that $B \cap B_i = W_i \in W$ and $\partial_\infty B_1 \cap \partial_\infty B_2$ is the set of poles of $B$.

*Proof.* Suppose $B_1, B_2 \in B$ are distinct blocks, $\xi \in \partial_\infty B_1 \cap \partial_\infty B_2$, and $W \in W$ is a wall separating $B_1$ from $B_2$. Choose basepoints $b_i \in B_i$, $w \in W$. If $x_k \in b_i \xi$ is a sequence tending to infinity, and $y_k \in b_\xi$ is a sequence with $d(y_k, x_k) < C$, then we can find a $z_k \in x_k y_k \cap W$ since $W$ separates $B_1$ from $B_2$. Therefore $\overline{wz_k} \subset W$ converges, and the limit ray $\overline{wx}$ lies in $W$. Hence $\xi \in \partial_\infty W$.

Note that if $W_1, W_2 \subset B \in B$, then $\partial_\infty W_1 \cap \partial_\infty W_2$ is just the set of poles of $B$; and $\xi \in \partial_\infty X$ cannot be a pole of two adjacent blocks simultaneously.

The lemma follows, since $\partial_\infty B_1 \cap \partial_\infty B_2 \neq \emptyset$ now implies that the combinatorial distance between $B_1$ and $B_2$ in $\text{Nerve}(B)$ is $\leq 2$. □

**Lemma 4** Suppose $\xi$ lies on the ideal boundary of a block $B \in B$, and assume $\xi$ is not a pole of any block other than $B$. Then the path component of $\xi$ in a suitable neighborhood $\Omega$ of $\xi$ is contained in $\partial_\infty B$.

*Proof.* Case I: $\xi \in \partial_\infty B$ is a pole of $B$. Choose $p \in B \setminus \bigcup_{W \in W} W$. Recall (see section 1.3) that $\alpha$ is the angle between singular geodesics of adjacent blocks lying in the common wall, so $\alpha$ is the minimum Tits angle between $\xi$ and any pole of a block adjacent to $B$. Let $\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\alpha}{2}\}$, where $\angle_p(\xi, \xi')$ is the angle between the initial velocities of the two rays $\overline{p\xi}, \overline{p\xi'}$. We define an exit from $B$ to be a singular geodesic $E \subset B$ of a block adjacent to $B$. A ray $\overline{p\xi'}$ exits from $B$ via $E$ if $\overline{p\xi'} \cap B$ is a geodesic segment ending at $E$, and the ray $\overline{p\xi'}$ continues into the block containing $E$. For each exit $E$ from $B$, let $\Omega_E$ be the set of $\xi' \in \Omega$ such that $\overline{p\xi'}$ exits $B$ via $E$. □
Sublemma 5 $\Omega_E$ is an open and closed subset of $\Omega$.

Proof. Openness. If $\xi' \in \Omega_E$, then $\overline{p\xi'} \cap B$ is a segment ending at some $e \in E$, and $\overline{p\xi'}$ enters the block $B'$ adjacent to $B$ which contains $E$. But then any sufficiently nearby ray $\overline{p\xi'}$ also leaves $B$ at a point close to $e$; clearly this point must lie on $E$. Therefore $\Omega_E$ is open in $\partial_\infty X$.

Closedness. Let $E' \subseteq E$ be the set of “exit points” for elements of $\Omega_E$: the endpoints of segments $\overline{p\xi'} \cap B$, where $\xi' \in \Omega_E$. $E'$ is bounded, for otherwise we could find a sequence $e_k \in E'$ with $\lim_{k \to \infty} d(e_k, p) = \infty$, and get a limit ray $\overline{pe_\infty} \subset B$ with $e_\infty \in \partial_\infty E \subset \partial_\infty B \cap \partial_\infty B'$, and $\angle_p(e_\infty, e_\infty) \leq \frac{\alpha}{2}$; this is absurd since $e_\infty$ is a pole of $B'$ and so $\angle_p(e_\infty, \xi) = \angle_{\text{Tits}}(e_\infty, \xi) \geq \alpha$. Now suppose $\xi'_k \in \Omega_E$ and $\lim_{k \to \infty} \xi'_k = \xi'_\infty \in \Omega$. We have, after passing to a subsequence if necessary, that $\overline{p\xi'_k} \cap B = \overline{p\xi'_\infty}$ where $e_k \in E$ and $\lim_{k \to \infty} e_k = e_\infty \in E$. Then $\overline{p\xi'_\infty} \cap B$ contains $\overline{pe_\infty}$ and clearly we have $\xi'_\infty \in \Omega$. 

It follows that the connected (or path) component of $\xi$ in $\Omega$ is contained in $\partial_\infty B$, since any subset $C \subseteq \Omega$ containing $\xi$ and intersecting $\Omega_E$ admits a separation $C = (C \cap \Omega_E) \cup (C \setminus \Omega_E)$ into open subsets of $C$, and any $\xi' \in \Omega \setminus \partial_\infty B$ lies in $\Omega_E$ for some $E$.

Case II: $\xi \in \partial_\infty W$ where $W$ is the wall separating two adjacent blocks $B_1$, $B_2$, and $\xi$ is not a pole. Pick $p \in W$ not lying on a singular geodesic. Let $\psi$ be the minimum Tits distance between $\xi$ and a pole of $B_i$, $i = 1, 2$, and set

$$\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\psi}{2}\}.$$

Let $E$ be a singular geodesic of $B_1$ or $B_2$ which is contained in $W$. We say that the ray $\overline{p\xi'}$ exits $W$ via $E$ if $\overline{p\xi'} \cap W$ ends at a point in $E$, and $\overline{p\xi'}$ then immediately enters the block corresponding to $E$. Let $\Omega_E$ be the set of $\xi' \in \Omega$ so that $\overline{p\xi'}$ exits $W$ via $E$. One checks as in case I that $\Omega_E$ is closed and open in $\Omega$, so we conclude that the connected component of $\xi$ in $\Omega$ is contained in $\partial_\infty W$.

Case III: $\xi \in \partial_\infty B$ does not lie in the boundary of any block other than $B$. Let $\phi$ be the minimum Tits angle between $\xi$ and a pole of $B$, and set

$$\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\phi}{2}\}.$$

Pick $p \in B \setminus \cup_{W \in \Gamma} W$. Since $\xi$ is not a pole of $B$, the ray $\overline{p\xi}$ determines a half-plane $H \subset B$, the intersection of the flat planes in $B$ containing it. Let $B'_1$ be the collection of blocks adjacent to $B$. If $B' \in B'_1$ then $B' \cap H (= \overline{W \cap H}$ where $W = B \cap B'$ is the wall between $B$ and $B'$) is either empty, a singular geodesic of $B$, or a flat strip with finite width bounded by singular geodesics, for otherwise we would have $\xi \in \partial_\infty B'$. Removing the singular geodesics and $\cup_{B' \in B'_1} B'$ from $H$, we get a subset $H^0$ whose connected components are a countably infinite collection of open strips. If $S \subset H^0$ is such a strip, we let $\Omega_S$ be the set of $\xi' \in \Omega$ so that $\overline{p\xi'} \cap S \neq \emptyset$. As in cases I and II, $\Omega_S$ is closed and open in $\Omega$. This forces the connected component of $\xi$ in $\Omega$ to be contained in $\partial_\infty H \subset \partial_\infty B$, as desired. 

□
1.7 Vertices and safe paths

We say that \( \xi \in \partial_\infty X \) is a **vertex** if there is a neighborhood \( U \) of \( \xi \) such that the path component of \( \xi \) in \( U \) is homeomorphic to the cone over a Cantor set, with \( \xi \) corresponding to the vertex of the cone. By Lemma 4 a pole of any block is a vertex (a priori there may be other vertices in \( \partial_\infty X \)).

A path \( c : [0, 1] \to \partial_\infty X \) is **safe** if \( c(t) \) is a vertex for only finitely many \( t \in [0, 1] \).

**Lemma 6** \( \cup_{B \in \mathcal{B}} \partial_\infty B \) is a safe path component of \( \partial_\infty X \).

**Proof.** First note that if \( c : [0, 1] \to \partial_\infty X \) is a path, \( c(t) \) is not a vertex when \( t \in (0, 1) \), \( B \in \mathcal{B} \), and \( c(0) \in \partial_\infty B \) is not a pole of any block other than \( B \), then \( c([0, 1]) \subset \partial_\infty B \).

This follows from Lemma 4, the fact that \( \partial_\infty B \) is closed in \( \partial_\infty X \), and a continuity argument.

Now if \( B_0 \in \mathcal{B} \), \( c : [0, 1] \to \partial_\infty X \) is a safe path starting in \( \partial_\infty B_0 \), and \( 0 = t_0 < t_2 \ldots < t_k = 1 \) are chosen so that \( c(t) \) is a vertex only if \( t = t_i \) for some \( i \), then one proves by induction on \( i \) that the intervals \([t_{i-1}, t_i]\) are mapped into \( \cup_{B \in \mathcal{B}} \partial_\infty B \).

**Lemma 7** Pick \( B_0 \in \mathcal{B} \) and \( p \in B_0 \setminus \cup_{W \in \mathcal{W}} W \). Let \( c : [0, 1] \to \partial_\infty X \) be a path, and suppose \( c(0) \) has an infinite \( p \)-itinerary. Then either \( c(t) \) has the same \( p \)-itinerary as \( c(0) \) for all \( t \in I \), or there is a \( t \in I \) so that \( c(t) \) has finite itinerary.

**Proof.** Suppose \( \xi_k \in \partial_\infty X \) is a sequence with \( \lim_{k \to \infty} \xi_k = \xi \in \partial_\infty X \), and a certain block \( B \) is in the itinerary of \( \overline{p\xi_k} \) for every \( k \). Then either

1. \( \text{Itin}(\xi) \) contains \( B \)

or

2. \( \text{Itin}(\xi) \) is finite and only contains blocks lying between \( B_0 \) and \( B \).

To see this, suppose \( B' \) is in \( \text{Itin}(\xi) \) and \( x \in \overline{p\xi} \cap \text{Int}(B') \). Then \( x = \lim_{j \to \infty} x_j \) where \( x_j \in \overline{p\xi_j} \cap \text{Int}(B') \) for sufficiently large \( j \), so \( B' \) is in \( \text{Itin}(\xi_j) \) for sufficiently large \( j \). This means that \( B' \) lies between \( B_0 \) and \( B \), for otherwise \( B \) would have to lie between \( B_0 \) and \( B' \), forcing \( B \in \text{Itin}(\xi) \).

The lemma now follows, since if \( B \) is in \( \text{Itin}(c(0)) \) but not in \( \text{Itin}(c(t)) \) for all \( t \in [0, 1] \), then setting \( t_0 := \inf \{ t \mid B \notin \text{Itin}(c(t)) \} \) we get a ray \( \overline{pc(t_0)} \) with finite itinerary by the reasoning of the preceding paragraph.

**Corollary 8** There is a unique safe path component of \( \partial_\infty X \) which is dense, namely \( \cup_{B \in \mathcal{B}} \partial_\infty B \).

**Proof.** By Lemma 6 we know that \( \cup_{B \in \mathcal{B}} \partial_\infty B \) forms a safe path component. \( \cup_{B \in \mathcal{B}} \partial_\infty B \) is dense in \( \partial_\infty X \) since any initial segment \( \overline{p\xi} \) of a ray \( \overline{p\xi} \) may be continued as a ray \( \overline{p\xi} = \overline{px} \cup \overline{\xi x} \) where the continuation \( \overline{\xi x} \) lies in a block (one of at most two) containing \( x \).

By Lemma 7, if \( \xi \in \partial_\infty X \) has an infinite \( p \)-itinerary, then any safe path starting at \( \xi \) consists of points with the same \( p \)-itinerary. Clearly the collection of points with a given \( p \)-itinerary isn’t dense in \( \partial_\infty X \). The corollary follows.
1.8 Detecting block boundaries

Call an arc $I \subset \bigcup_{B \in \mathcal{B}} \partial_{\infty} B$ an edge if its endpoints are both vertices, but no interior point of $I$ is vertex of $\partial_{\infty} X$. Edges are contained in the boundary of a single block $B \in \mathcal{B}$ (see the proof of Lemma 6). Clearly the endpoints of an edge $I \subset \bigcup_{B \in \mathcal{B}} \partial_{\infty} B$ are either the poles of a single block, or $I \subset \partial_{\infty} W$ where $W = B_1 \cap B_2$ and the endpoints of $I$ are poles of $B_1$ and $B_2$. So two points in $\bigcup_{B \in \mathcal{B}} \partial_{\infty} B$ are the poles of a single block (resp. adjacent blocks) iff they are the endpoints of more than one edge (resp. a unique edge). A subset of $\bigcup_{B \in \mathcal{B}} \partial_{\infty} B$ is the boundary of a block $B \in \mathcal{B}$ iff it is the union of all edges intersecting the poles of $B$.

1.9 Limiting behavior of poles

Pick $B \in \mathcal{B}$, and consider the set $\mathcal{P}$ of poles of blocks adjacent to $B$. If $\eta \in \partial_{\infty} B$ is a pole of $B$, then we have $\angle_{\text{Tits}}(\xi, \eta) \in \{\alpha, \pi - \alpha\}$ for every $\xi \in \mathcal{P}$. Let $\mathcal{P}_\alpha := \{\xi \in \mathcal{P} | \angle_{\text{Tits}}(\xi, \eta) = \alpha\}$, and $\mathcal{P}_{\pi - \alpha} := \{\xi \in \mathcal{P} | \angle_{\text{Tits}}(\xi, \eta) = \pi - \alpha\}$. Call each arc of $\partial_{\infty} B$ joining the poles of $B$ a longitude.

**Lemma 9** Each longitude of $\partial_{\infty} B$ intersects $\mathcal{P}_\alpha$ (resp. $\mathcal{P}_{\pi - \alpha}$) in a single point $\xi$ with $\angle_{\text{Tits}}(\xi, \eta) = \alpha$ (resp $\angle_{\text{Tits}}(\xi, \eta) = \pi - \alpha$).

**Proof.** Pick $p \in B$, $\xi \in \partial_{\infty} B$ with $\angle_{\text{Tits}}(\xi, \eta) = \alpha$. Any initial segment $\overline{px}$ of the ray $\overline{px}$ may be extended to a segment $\overline{pxy} = \overline{px} \cup \overline{xy}$ so that $\overline{xy} \cap W = \{y\}$ for some wall $W \subset B$. Then $\overline{xy}$ may be extended as a ray $\overline{px'} = \overline{p}y = \overline{y}x'$ where $y' \in W$ and $x' \in \mathcal{P}_\alpha$. Therefore $\xi \in \mathcal{P}_\alpha$. Since $\angle_{\text{Tits}}(\cdot, \eta)$ is a continuous function on $\partial_{\infty} B$, each longitude intersects $\mathcal{P}_\alpha$ in a single point. Similar reasoning applies to $\mathcal{P}_{\pi - \alpha}$. □

From the lemma we see that any longitude $l$ of $\partial_{\infty} B$ intersects $\mathcal{P}$ in two points if $\alpha < \frac{\pi}{2}$ and one point if $\alpha = \frac{\pi}{2}$.

1.10 Distinguishing Leeb complexes

Let $\tilde{X}_1$ be a Leeb complex with $\alpha < \frac{\pi}{2}$, and let $\tilde{X}_2$ be a Leeb complex with $\alpha = \frac{\pi}{2}$. Let $X_1$ and $X_2$ be their respective universal covers. A homeomorphism $f : \partial_{\infty} X_1 \to \partial_{\infty} X_2$ would carry safe path components to safe path components, block boundaries to block boundaries (Corollary 8 and section 1.8), poles to poles, and longitudes to longitudes. But then section 1.9 gives a contradiction.

References


