In the one variable case $z = f(y)$ and $y = g(x)$ then $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$.

When there are two independent variables, say $w = f(x, y)$ is differentiable and where both $x$ and $y$ are differentiable functions of the same variable $t$ then $w$ is a function of $t$ and

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Show tree diagram.
Problem: Compute $\frac{dw}{dt}$ two ways when $w = x^2 + xy$, $x = \cos(t)$, and $y = \sin(t)$. 
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What is $\frac{dw}{dt}(\frac{\pi}{2})$?

Do tree diagram for $w=f(x,y,z)$ where $x,y,z$ are functions of $t$. 
**Problem:** Compute \( \frac{dw}{dt} \) two ways when \( w = x^2 + xy, \ x = \cos(t), \) and \( y = \sin(t). \)

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What do you do if the intermediate variables are also functions of two variables? Say \( w = f(x,y,z) \) where each of \( x,y,z \) are functions of \( r \) and \( \theta. \) This makes \( w \) a function of \( r \) and \( \theta. \)
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**Problem:** Compute $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of $r$ and $s$ when $w = x + y - z^2$, $x = rs$, $y = r + s$, and $z = e^{rs}$. 

Christopher Croke
Calculus 115
If $F(x, y) = 0$ defines $y$ implicitly as a function of $x$, that is $y = h(x)$, and if $F$ is differentiable then:

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Problem: Find $\frac{dy}{dx}$ if $x^2 + y^2 x + \sin(y) = 0$. 

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$$x(s) = u_1 s + x_0 \quad y(s) = u_2 s + y_0.$$
Directional Derivatives

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The directional derivative in the direction \( \vec{u} \) is thus

\[
\left( \frac{df}{ds} \right)_{\vec{u}, (x_0, y_0)} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \equiv (D_{\vec{u}} f)_{(x_0, y_0)}.
\]
Definition:

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(D_{\vec{u}} f)_{(x_0,y_0)} = \lim_{s \to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}.
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We call the vector \( f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} \) the gradient of \( f \) at \((x_0, y_0)\) and we denote it \( \nabla f \).
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**Problem:** Compute $D_{\vec{u}} f$ where $f(x, y) = e^{xy} + x^2$ at the point $(1, 0)$ in the direction $\vec{u} = \frac{2}{\sqrt{13}} \vec{i} - \frac{3}{\sqrt{13}} \vec{j}$.
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If $\theta$ is the angle between $\nabla f$ and $\vec{u}$ then

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos(\theta) = |\nabla f| \cos(\theta).$$

This means that $f$ increases most rapidly in the direction of $\nabla f$.

(And least rapidly in the direction $-\nabla f$.)

Another conclusion is that if $\vec{u} \perp \nabla f$ then $D_{\vec{u}}f = 0$. 

Christopher Croke
Calculus 115
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Now \(f(x, y)\) does not change along the level curves \(c = f(x, y)\) for any constant \(c\). If we write the curve as \((x(t), y(t))\) and use the chain rule we see:

\[
\frac{d}{dt} f(x(t), y(t)) = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} = 0.
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**Problem:** Find tangent line to the curve

\[x + \sin(y) + e^{xy} = 2\]

at the point \((1, 0)\).
Let's look at three independent variables, i.e. \( f(x, y, z) \). Then

\[
\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}.
\]

This means that \( \nabla f \) is perpendicular to the level surfaces of \( f \). (i.e. it is normal to all curves in the level surface.)
Three variables

Let's look at three independent variables, i.e. \( f(x, y, z) \). Then

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The **Tangent Plane** at $(x_0, y_0, z_0)$ to the level surface $f(x, y, z) = c$ is the plane through the point $(x_0, y_0, z_0)$ normal to $\nabla f(x_0, y_0, z_0)$. 
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Thus the equation is:

$$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0.$$
The **Normal Line** of the surface at \((x_0, y_0, z_0)\) is the line through \((x_0, y_0, z_0)\) which is parallel to \(\nabla f\).
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**Problem:** Find the tangent plane and normal line to the graph of \(z = f(x, y) = x^2 - 3xy\) at the point \((1, 2, -5)\).