MATH 114 ASSIGNMENT 11, FALL 2015

Due in class on Monday, November 23.

Summary of § 16.3. Let $\vec{F}$ be a smooth vector field in a connected open domain D in $\mathbb{R}^3$. (The case of a region in $\mathbb{R}^2$ is similar but easier.) The following statements about $\vec{F}$ are equivalent.

1. $\vec{F}$ is *conservative* on D, i.e. for any two connected oriented piecewise smooth curves C_1, C_2 in D with the same beginning point and the same end point, $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

   [Note: another terminology, with the same meaning, is the that line integrals of $\vec{F}$ of curves in D are *path independent*.]

2. There exists a smooth function f on D such that $\mathrm{grad}(f) = \vec{F}$.

   [Note. Such a function f is called a *potential* of $\vec{F}$ on D. Any two potentials of $\vec{F}$ differ by a constant.]

Properties of conservative vector fields on D are

3. If f is a potential of $\vec{F}$, and C is an oriented piecewise smooth curve in D from point P to point Q, then $\int_C \vec{F}\cdot d\vec{r} = \int_{\partial C} f = f(Q) - f(P)$.

4. If $\vec{F}$ is conservative, then $\mathrm{curl}(\vec{F}) = \vec{0}$.

Property 4 above says that conservative vector fields have curl $\vec{0}$. The converse is *not* true in general. For instance the curl of the vector field $\vec{G} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$ is $\vec{0}$, but the line integral $\oint_C \vec{G}\cdot d\vec{r}$ over an orieneted connected closed curve C in the complement of the z-axis in $\mathbb{R}^3$ may not be zero: it is equal to 2π times the number of time C winds about the z-axis "in the counterclockwise direction". (The "number of times" is counted with signs. For instance if the oriented curve C is parametrized by $\vec{r}(t) = 2\cos(2\pi t)\vec{i} - 3\sin(2\pi t)\vec{j} + t(1-t)$, $t \in [0,1]$, then the *winding number* of C with respect to the z-axis is -1.)

Properties of vector fields with zero curl.

5. Suppose that $\vec{F}$ is a smooth vector field on D such that $\mathrm{curl}(\vec{F}) = \vec{0}$. Then For every pont P of D, there exists an open subset $U \subset D$ and a smooth function f on U such that $\vec{F}|_U = \mathrm{grad}(U)$. In other words $\vec{F}$ is *locally* a gradient.

6. If D is *simply connected*, and $\vec{F}$ is a smooth vector field on D with $\mathrm{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is a gradient

The intuitive meaning that an open domain D in $\mathbb{R}^3$ is *simply connected* is that

> every closed curve in D can be continuously deformed in D to a point in D, i.e. a curve parametrize by a constant function $\vec{r}$.

We will not give a precise definition here, but only give a few examples:

(a) the whole space $\mathbb{R}^3$,
(b) any open ball,
(c) any open ellipsoid,
(d) any open cube,
(e) $\mathbb{R}^3$ with several points removed,

(f) the region inside an ellipsoid E but outside a smaller ellipsoid E_1 inside E, and

(g) an open ball or an open ellipsoid with several points removed

are all simply connected, while

(h) $\mathbb{R}^3$ with several lines removed

(i) solid tori,

(j) $\{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 \leq 100,\ y^2+z^2 \geq 1\}$,

(k) $\{(x,y,z) \in \mathbb{R}^3 | 3 \leq x^2+z^2 \leq 10\}$ and

(l) the solid obtained by rotating the ring-shaped region $\{(x,z) \in \mathbb{R}^2 | 1 \leq (x-10)^2+z^2 \leq 2\}$ on the (x,z)-plane about the z-axis

are *not* simply connected.

If you are interested to find out more about these, you can search the following keywords: closed differential forms, exact differential forms, de Rham cohomlogy, fundamental group, simply connected.

Part 1. Read Thomas 16.3, 16.4, 16.7, 16.8, §4 of Notes on vector calculus. and the short summary of § 16.3 above.

Part 2. Do and write up the following problems, from Thomas and old final exams.

16.3 Exercises 22, 32

16.4 Exercise 39

16.7 Exercise 27, 28

16.8 Exercose 24

S14 Math 114 final exam, May 5, 2014, problem 9.

S14 Math 114 final exam, May 5, 2014, problem 14.

F13 Math 114 final exam, December 7, 2013, problem 12.

F13 Math 114 final exam, December 7, 2013, problem 14.

Part 3. Extra credit problems

A. (a) Give an example of a vector field $\vec{F}$ which is continuously differentiable on $\mathbb{R}^3 - \{(0,0,z) \mid z \in \mathbb{R}\}$ (but possibly has singularities on the z-axis), such that

$$\mathrm{curl}(\vec{F}) = \vec{0} \quad \text{and} \quad \vec{F} \neq \mathrm{grad}(g)$$

for any differentiable function g on $\mathbb{R}^3 - \{(0,0,z) \mid z \in \mathbb{R}\}$.

(b) Give an example of a vector field $\vec{F}$ which is continuously differentiable on $\mathbb{R}^3 - \{(0,0,0)\}$ (but has a singularity at the origin), such that

$$\mathrm{div}(\vec{F}) = 0 \quad \text{and} \quad \vec{F} \neq \mathrm{curl}(\vec{G})$$

for any differentiable vector field $\vec{G}$ on $\mathbb{R}^3 - \{(0,0,0)\}$. Please fully justify your answer.

B. Let E be the solid ellipsoid

$$E := \{(x,y,z) \in \mathbb{R}^3 \mid 2x^2 + xy + 2y^2 + 2z^2 \leq 7\}$$

and let S be the boundary surface of E:

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 2x^2 + xy + 2y^2 + 2z^2 = 7\}.$$

Orient S by the unit normal vector field $\vec{N}$ on S pointing away from E at every point of S. Compute the oriented surface integral

$$\iint_S \operatorname{curl}\left(\frac{1}{(x^2+y^2+z^2)^2}(x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k})\right) \cdot \vec{N}\,\mathrm{d}A\,.$$