13.6 Kepler's first law

$\vec{r} = r \cdot \vec{u}_r$ polar coord.

$\vec{u}_r = \cos\theta \, \vec{i} + \sin\theta \, \vec{j}$

$\vec{u}_\theta = -\sin\theta \, \vec{i} + \cos\theta \, \vec{j}$

$\frac{d}{dt}\vec{u}_r = \dot\theta \, \vec{u}_\theta, \quad \frac{d}{dt}\vec{u}_\theta = -\dot\theta \, \vec{u}_r$

$$\frac{d\vec{r}}{dt} = \dot r \, \vec{u}_r + r\dot\theta \, \vec{u}_\theta$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\ddot r - r(\dot\theta)^2\right)\vec{u}_r + \left(2\dot r\dot\theta + r\ddot\theta\right)\vec{u}_\theta$$

Newton's law: $\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\vec{u}_r$

G = gravitational constant

M = mass of Sun

center of polar coord = location of Sun

means

$$\begin{cases} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} & -(1) \\ 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 & -(2) \end{cases}$$

(2) $\Rightarrow$ $\ell := r^2\frac{d\theta}{dt}$ is constant (i.e. independent of t)

Note: ℓ = (angular momentum of planet)/(mass of planet)

Key to integrate the differential equation (1):

Use the relation $\frac{d}{dt} = \frac{\ell}{r^2}\frac{d}{d\theta} = \ell\rho^2\frac{d}{d\theta}$ $-(3)$ where $\rho = \frac{1}{r}$

to turn (1) into a differential equation for $\rho := \frac{1}{r}$

with respect to the variable θ

Have: $\frac{dr}{dt} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = -\ell\frac{d\rho}{d\theta}$ (by (3))

$\frac{d^2r}{dt^2} = -\ell^2\rho^2\frac{d^2\rho}{d\theta^2}$ (use (3) again)

The differential equation (2) becomes

$$-\ell^2\rho^2\frac{d^2\rho}{d\theta^2} - \ell^2\rho^3 = -\rho^2 GM, \text{ i.e.}$$

$$\frac{d^2\rho}{d\theta^2} + \rho = \frac{GM}{\ell^2} \quad -(4)$$

where $\rho := \frac{1}{r}$, M = mass of the Sun

$\ell = r^2\frac{d\theta}{dt}$ = a constant

(part of Kepler's 2nd law)

The general solution of the linear ODE (4) is

$$\rho = \frac{GM}{\ell^2} + A\cdot\cos(\theta + B)$$ where A, B are (arbitrary) constants

We may assume that $B = 0$ (adjust/rotate the polar coord system by an angle B)

The resulting curve $r = \frac{1}{\frac{GM}{\ell^2}(1 + e\cos\theta)}$ $\left(e = \frac{A}{GM/\ell^2}\right)$ is an ellipse

Kepler's third law: $T^2 \underset{\text{proportional to}}{\sim} a^3$ where T = period of the planet

(The constant of proportionality is independent of the planet) a = semi-major axis of the elliptical orbit

This is a routine verification, but one needs to recall some basics of the geometry of ellipse:

(5) $$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

where

(6) $a(1-e^2) = GM/\ell^2$

a = semi-major axis

b = semi-minor axis

$e^2 = (a^2-b^2)/a^2$, e = eccentricity (terminology)

area of ellipse $= \pi a b$

We need to figure out what the period T is

Main clue: the rate of change of the area swept out by the planet (with respect to the Sun) is $\frac{1}{2} r^2 \frac{d\theta}{dt} = \underbrace{\text{a constant}}_{\text{but depends on the planet}} = \frac{\ell}{2}$

$\Rightarrow$ $T = 2\pi a b/\ell$ — ( )

Compute $$T^2 = \frac{4\pi^2 a^2 b^2}{\ell^2} = \frac{4\pi\, a^2 b^2}{GM/a(1-e^2)} = \frac{4\pi a^2 b^2}{GM\, a^{-1}\cdot b^2} = \frac{4\pi}{GM}\cdot a^3$$

i.e. $\frac{T^2}{a^3} = \frac{4\pi}{GM}$; this proportionality constant depends only on the mass of the Sun.