NOTES ON VECTOR CALCULUS

We will concentrate on the *fundamental theorem of calculus* for curves, surfaces and solids in $\mathbb{R}^3$. These are equalities of *signed* integrals, of the form

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha\,,$$

where M is an *oriented* n-dimensional geometric body, and α is an "integrand" for dimension $n-1$, ∂M is the boundary of M, and $\mathrm{d}\alpha$ is a *signed* derivative of α suitable for integration in dimension n. It is important that the dimensions match for both sides of the above equality:

- ∂M is the boundary of M, so the dimension of ∂M is $n-1$,
- α is an integrand for dimension $n-1$, and the signed derivative d is defined so that the dimension of $\mathrm{d}\alpha$ is 1 more than the dimension of α.
- On the left-hand-side of the equality the dimensions of ∂M and α are both $n-1$. On the right-hand-side of the equality, the dimesions of M and $\mathrm{d}\alpha$ are both n.

Signed integrals are designed so that nice cancellations happen when one performs integration by parts. The fundamental theorem of calculus is essentially integration by parts in higher dimensions, it holds because of these cancellations.

Cancellation also happens for the signed derivatives, so that if one takes the signed derivatives twice consecutively one gets 0; see 1.2 for details.

Since M is contained in $\mathbb{R}^3$, we have $n = 0, 1, 2$ or 3. The case when $n = 2$ is called Gauss' (or divergence) theorem. The case when $n = 1$ is called Stokes' theorem. If $n = 1$ and M is contained in $\mathbb{R}^2$, then Stokes' theorem specializes to Green's theorem.

§1. Signed integrands and their derivatives

(1.1) In the traditional notation used in standard textbooks in engineering mathematics, integrands in dimensions 0 and 3 are invariably expressed in terms of scalar valued functions $f(x,y,z)$. For instance triple integrals for domains D in $\mathbb{R}^3$ look like

$$\iiint_D f(x,y,z)\,dx\,dy\,dz.$$

In contrast, integrands in dimensions 1 are 2 come in two flavors: those with signs and those without signs. The *signed* integrals are expressed via vector valued functions, or *vector fields*

$$\vec{F}(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$$

with three components. The signed integrands in dimension 1 have the form

$$\vec{F}\cdot\vec{T}\,ds \;=\; \vec{F}\cdot d\vec{x},$$

while the signed integrands in dimension 2 have the form

$$\vec{F}\cdot\vec{N}\,d\sigma \;=\; \vec{F}\cdot d\vec{S}.$$

In the two displayed formulae above, the left hand side is the notation used in the textbook by Thomas, while the right hand side is the notation used in many other books; the two notation systems are equally popular.

REMARK (a) The standard one-dimensional integral

$$\int_a^b f(x)\,dx$$

is a *signed* integral: $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$

(b) There is an alternative system of notation, which generalizes to higher dimensions. Unfortunately this superior approach to the fundamental theorem of calculus has not been widely adopted by scientists and engineers (yet), so we will not use it in class. In this system an one-dimensional integrand for a vector field $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ is written as

$$f_1\,\mathrm{d}x + f_2\,\mathrm{d}y + f_3\,\mathrm{d}z\,,$$

while the two-dimensional integrand corresponding to $\vec{F}$ is written as

$$f_1\,\mathrm{d}y \wedge \mathrm{d}z + f_2\,\mathrm{d}z \wedge \mathrm{d}x + f_3\,\mathrm{d}x \wedge \mathrm{d}y\,.$$

You might wonder what these strange creatures $\mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}z \wedge \mathrm{d}x$ and $\mathrm{d}x \wedge \mathrm{d}y$ are. They are analogues of the "dx" in a typical integral $\int_a^b g(x)\,\mathrm{d}x$ or the "$\mathrm{d}x\,\mathrm{d}y$" in a typical double integral $\int_D h(x,y)\,\mathrm{d}x\,\mathrm{d}y$; the symbol $\wedge$ emphasizes that the corresponding integrals are *signed*.

(1.2) Identities for signed derivatives

(1.2.1) The derivative involved in the fundamental theorem of calculus, from an n-dimensional integrand to an $n+1$-dimensional integrand, is given in the following table. These derivatives are defined so that

$$\mathrm{d}(\mathrm{d}\alpha) = 0$$

for all α. For integrands of dimensions $0, 1, 2$ in $\mathbb{R}^3$, the signed derivatives correspond to the three operators gradient, curl and divergence respectively.

n	$\alpha \mapsto \mathrm{d}\alpha$	$\mathrm{d}(\mathrm{d}\alpha) = 0$
0	$f \mapsto \mathrm{grad}(f)$	
1	$\vec{F} \mapsto \mathrm{curl}(\vec{F})$	$\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$
2	$\vec{G} \mapsto \mathrm{div}(\vec{G})$	$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$

Integrands for dimension 4 or above on $\mathbb{R}^3$ are trivial–they are all 0. So we don't have other signed derivatives on $\mathbb{R}^3$ other than gradient, curl and divergence. For the same reason, $\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$ and $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$ are the only two cases of the general identity $\mathrm{d}(\mathrm{d}\alpha) = 0$ for $\mathbb{R}^3$.

REMARK For spaces of dimension 4 or above, the signed integrands have more components. For instance on $\mathbb{R}^5$, integrands for dimensions 2 and 3 have 10 components, integrands for dimensions 1 and 4 have 5 components, while integrands for dimensions 0 and 5 have just 1 component. Thus the signed derivative for 2-dimensional integrands will send a vector field with 5 components to a vector field with 10 components, while the signed derivative for 3-dimensional integrands will send a vector field with 10 components to a vector field with 10 components.

(1.2.2) Poincaré's Lemma

If an integrand α in dimension $n \geq 1$ satisfies $\mathrm{d}\alpha = 0$, it is not necessarily true that there existgs an integrand β in dimension $n-1$ such that $\mathrm{d}\beta = \alpha$. However a *local form* holds: if $\mathrm{d}\alpha = 0$ then for every point where α is smooth, there exists an $(n-1)$-dimensional integrand β ia an open neighborhood of this point such that $\alpha = \mathrm{d}\beta$ in this neighborhood.

- (n=1) Suppose that $\vec{F}$ is a smooth vector field on an open subset U of $\mathbb{R}^3$ such that $\mathrm{curl}(\vec{F}) = 0$, then for every point P of U, there exists an open neighborhood V of P and a smooth function f on V such that $\mathrm{grad}(f) = \vec{F}$ on V.

- (n=2) Suppose that $\vec{G}$ is a smooth vector field on an open subset U of $\mathbb{R}^3$ such that $\mathrm{div}(\vec{G}) = 0$, then for every point P of U, there exists an open neighborhood V of P and a smooth vector field G on V such that $\mathrm{curl}(F) = \vec{G}$ on V

§2. Orientation of curves and surfaces in $\mathbb{R}^3$

(2.1) How to orient a curve

An *orientation* of a smooth curve C is (determined by) a continuous unit tangent vector field, i.e. a tangent vector field on C with lenght 1 at every point of C. Note that every connected smooth curve C has exactly 2 orientations.

(2.1.1) The sign of a parametrization of an oriented curve

Let $\vec{r} : [a,b] \to C$ be a parametrization of (a piece of) an oriented curve $(C, \vec{T})$. The *sign* of such an parametrization is the function defined by

$$\mathrm{sign}(C, \vec{T}, \vec{r})(t) = \begin{cases} 1 & \text{if } \frac{\partial \vec{r}}{\partial t}(t) \text{ is a positive multiple of } \vec{T}(t) \\ 0 & \text{if } \frac{\partial \vec{r}}{\partial t}(t) = \vec{0} \\ -1 & \text{if } \frac{\partial \vec{r}}{\partial t}(t) \text{ is a positive multiple of } \vec{T}(t) \end{cases} \tag{1}$$

If $\mathrm{sign}(C, \vec{T}, \vec{r}) = 1$ throughout the interval $[a,b]$, we say that that parametrization $\vec{r}$ is *compatible* with the orientation of C.

(2.1.2) Remark Usually the derivative $\frac{\partial \vec{r}}{\partial t}(t) \neq \vec{0}$ for all $t \in [a,b]$; if this is the case then $\mathrm{sign}(C, \vec{T}, \vec{r})$ is a constant function on the interval $[a,b]$, equal to either 1 or -1.

(2.2) Orientation of surfaces in $\mathbb{R}^3$

An *orientation* of a surface S in $\mathbb{R}^3$ is given by a continuous *normal* vector field $\vec{N}$ of unit length on S. Put it in another way, you assign an orientation to a surface S by picking a "preferred side" of the surface in a continuous fashion (if that is possible).[1] If S is connected and there is an orientation of S, then there are exactly two orientations.

(2.2.1) (a) In practice all you need to do is to pick a vector field on S which is *not* tangent to S at every point of S: you can subtract suitable tangent components to produce a vector field on S with is non-zero and normal to S at every point of S, then scale this normal vector field to produce a unit normal vector field.

[1]Continuity means that you are not allowed to abruptly change the preferred side. Some surfaces, such as the Möbius band, are one-sided and not orientable.

(b) If the surface S is parametrized by a smooth vector-valued function $\vec{r}: D \to \mathbb{R}^3$ on a region $D \subset \mathbb{R}^2$ such that $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are not parallel at every point of D, then the two unit normal vector fields on S are

$$\pm \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}$$

(2.2.2) An alternative way to think of orientation of surfaces

Given an orientation $\vec{N}$ of a surface S in $\mathbb{R}^3$, one can decide for any *ordered pair* $(\vec{v}, \vec{w})$ of vectors on the tangent plane E to S at a point $\vec{a} \in S$ is *compatible with the given orientation*:

> $(\vec{v}, \vec{w})$ is compatible with the orientation given by the unit vector field N on S if $\vec{v} \times \vec{w}$ is a positive multiple of the vector $\vec{N}(a)$.

Note that if $(\vec{v}, \vec{w})$ is incompatible with the given orientation, then $(\vec{w}, \vec{v})$ is. If S is parametrized by a vector-valued function $\vec{r}: D \to S$ on a connected region D in the (u, v) plane, we say that this parametrization $\vec{r}$ is compatible with the orientation if the ordered pair $\left(\frac{\partial \vec{r}}{\partial u}(u,v), \frac{\partial \vec{r}}{\partial v}(u,v)\right)$ is compatible with $\vec{N}$ for every point of S.

Remark (a) When defining *oriented surface integrals* for an oriented surface, one uses parametrizations which are compatible with the given orientation S. (If you happened to have picked a parametrization which is incompatible with the orientation, you need to compensate by changing the sign.)

(b) We just saw that an orientation of surface S in $\mathbb{R}^3$ specifies among local coordinate systems of S s subfamily consisting of local coordinate systems which are compatible with the given orientation. This latter notion gives an alternative, and in many aspects more intrinsic description of the idea of orientation, because it generalizes to "higher dimensional surfaces".

(2.2.3) The sign of a parametrization of an oriented surface

Given an orientation $\vec{N}$ of a surface S in $\mathbb{R}^3$, for any parametrization $\vec{r}$ of S, we can define a function $\operatorname{sign}(S, \vec{N}, \vec{r})$ whose values are ± 1, given by

$$\operatorname{sign}(S, \vec{N}, \vec{r})(u,v) := \operatorname{sign}\left(\det\left(\vec{N}(\vec{r}(u,v)), \frac{\partial \vec{r}}{\partial u}(u,v), \frac{\partial \vec{r}}{\partial v}(u,v)\right)\right) \tag{2}$$

In other words it is the *sign* of the determinant of the 3×3 matrix whose three columns are the vector-valued functions $\vec{N}$, $\partial \vec{r}/\partial u$ and $\partial \vec{r}/\partial v$ on D; it is equal to ± 1 if the domain of definition $D \subset \mathbb{R}^2$ of the function $\vec{r}(u,v)$ is *connected*. In other words, $\operatorname{sign}(S, \vec{N}, \vec{r}) = 1$ if $\vec{N}$ is a positive multiple of $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$, while $\operatorname{sign}(S, \vec{N}, \vec{r}) = -1$ if $\vec{N}$ is a negative multiple of $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$. In the terminology of 2.2.2,

$$\operatorname{sign}(S, \vec{N}, \vec{r})(u,v) = \begin{cases} 1 & \text{if } \left(\frac{\partial \vec{r}}{\partial u}(u,v), \frac{\partial \vec{r}}{\partial v}(u,v)\right) \text{ is compatible with the orientation } \vec{N}(u,v) \\ -1 & \text{if } \left(\frac{\partial \vec{r}}{\partial u}(u,v), \frac{\partial \vec{r}}{\partial v}(u,v)\right) \text{ is incompatible with the orientation } \vec{N}(u,v) \end{cases}$$

This function $\operatorname{sign}(S, \vec{N}, \vec{r})(u,v)$ on S is a constant, equal to either 1 or -1, if the surface S is connected.

(2.3) How to orient the boundary of a solid in $\mathbb{R}^3$

Suppose that B is a domain in $\mathbb{R}^3$ with a smooth boundary ∂B.[2] Note that S may have several components. For instance if $B := \left\{(x,y,z) \in \mathbb{R}^3 | 1 \le x^2 + y^2 + z^2 \le 4\right\}$, then its boundary ∂B consists of two spheres, of radii 2 and 1 respectively.

[2]Here ∂B is the standard notation for "the boundary of B".

There is a standard convention for orienting the boundary surface ∂B of a 3-dimensional domain $B \subset \mathbb{R}^3$.

> Choose the unit normal vector field $\vec{N}$ on ∂B which points **away** from B at every point of the boundary surface ∂B.

The above way of orienting ∂B is important for the divergence theorem (also known as Gauss' theorem).

In the example $B := \left\{(x,y,z) \in \mathbb{R}^3 | 1 \leq x^2+y^2+z^2 \leq 4\right\}$, $\partial B = S_1 \cup S_2$, where S_1 is a sphere of radius 1 and S_2 is a sphere of radius 2. The unit normal vector field $\vec{N}$ is

$$\vec{N}(\vec{r}) = \begin{cases} -\vec{r} & \text{if } \vec{r} \in S_1 \\ \frac{1}{2}\vec{r} & \text{if } \vec{r} \in S_2 \end{cases}$$

(2.4) How to compatibly orient a surface S with boundary and it boundary ∂S

Suppose that S is smooth surface with boundary in $\mathbb{R}^3$, oriented by a continuous unit normal vector field $\vec{N}$. The boundary of S is a smooth curve ∂B, which may have a number of connected components. We want to define an orientation on the boundary curve ∂S, which is uniquely determined by the orientation $\vec{N}$ of S. In other words we want to specify a vector field $\vec{T}$ on ∂S, which is tangent to the curve ∂S and has lengh 1 at every point of ∂S. In fact we will produce *two* unit vector fields, $\vec{v}$ and $\vec{T}$, on the boundary curve ∂S.

(2.4.1) Definitions of vector fields $\vec{v}$ and $\vec{T}$ on ∂S

(a) We have a vector field $\vec{v}$ on ∂S, uniquely determined by the following conditions. This vector field $\vec{v}$ is completely determined by the geometry of the surface S; the orientation $\vec{N}$ of S is not involved in defining $\vec{v}$.

- $\| \vec{v}(P) \| = 1$ for every $P \in \partial S$;
- $\vec{v}(P)$ is tangent to the surface S at P, i.e. $\vec{v}(P) \cdot \vec{N}(P) = 0$, for every point $P \in \partial S$;
- $\vec{v}(P)$ is orthogonal to the tangent line of the curve ∂S at P, for every $P \in \partial S$;
- $\vec{v}(P)$ points **away** from S at every point $P \in \partial S$.

(b) The vector field $\vec{T}$ on ∂S is the unique tangent vector field of unit length on ∂S such that

$$\det(\vec{N}(P), \vec{v}(P), \vec{T}(P)) > 0 \qquad \text{for every point } P \in \partial S\,.$$

REMARK (a) Because the three unit vectors $\vec{N}(P)$, $\vec{v}(P)$ and $\vec{T}(P)$ are orthogonal to each other, this condition is equivalent to each of the following statements.

- $\det(\vec{N}(P), \vec{v}(P), \vec{T}(P)) = 1 \qquad$ for every point $P \in \partial S$
- $\vec{T}(P) = \vec{N}(P) \times \vec{v}(P) \qquad$ for every point $P \in \partial S$
- $\vec{N}(P) = \vec{v}(P) \times \vec{T}(P) \qquad$ for every point $P \in \partial S$
- $\vec{v}(N) = \vec{T}(P) \times \vec{N}(P) \qquad$ for every point $P \in \partial S$

Many people use the expression "$\vec{N}(P)$, $\vec{v}(P), \vec{T}(P)$ form a right-handed system" when any (hence each) of the above conditions holds. This is a common terminology which has nothing to do with people's hands.[3]

(b) To summarize: the geometry of the surface S and its boundary curve ∂S uniquely determines the unit vector field ν on ∂S, which is tangent to S, orthogonal to ∂S, and points **away from** ∂S at every point of S. The orientation $\vec{N}$ of S determines, with the help of $\vec{v}$, the orientation $\vec{T}$ of the boundary curve ∂S.

REMARK (Generalization to higher dimensions) Given an n-dimensional *oriented* geometric body M, whose boundary ∂M is a gemetric body of dimension $n-1$. Is there a way to give ∂M an orientation? Not surprisingly the answer is "yes". The idea is explained in the next paragraph.

We have a vector field $\vec{v}$ on the boundary ∂M which is tangent to M, normal to ∂M and pointing *away* from M, just as in the case of surfaces. The orientation on ∂M is specified so that the pair $(\vec{v}, \text{orientation of } \partial M)$ is compatible with the orientation of M, in the following sense: for every point P of ∂M, there is a local coordinate system $x_0, x_1, \ldots, x_{n-1}$ for M near P such that

- the boundary ∂M is given by $\{x_0 = 0\}$ near P,
- $x_1, \ldots, x_{n-1}$ is a local coordinate system near P compatible with the orientation of ∂M, and
- $x_0, x_1, \ldots, x_{n-1}$ is a local coordinate system near P compatible with the orientation of M.

(2.4.2) Here is an intuitive way to think about the orientation of the boundary curve ∂S induced by the orientation of the surface. Imagine that a tiny tiny creature stands on the preferred side of the surface S as given by the orientation $\vec{N}$, at a place of S pretty close to the boundary curve ∂S. Because the sense of scale of this creature is much smaller than ours, in its eyes the surface S looks completely flat and the boundary curve looks like a straight line. Suppose we want this creature to move forward along ∂S in the direction of the unit tangent vector field $\vec{T}$ on the curve ∂S, but this poor thing doesn't understand complicated concepts such as "the orientation of the boundary curve defined by the orientation of the surface". What should we say in the marching order?

Answer. Tell it to move forward on the surface S, keeping the boundary ∂S in sight, **to its right**.

(2.4.3) Boundary of a bounded region in the plane. A simple example is the case when the surface S lies on the (x,y)-plane, oriented by the constant normal vector field $\vec{k}$. The boundary ∂S is the disjoint union of a simple closed curve C_0, together with a finite number of simple closed curves $C_1, \ldots, C_m$ lying in the domain enclosed by C_0. In this case, the orientation of ∂S induced by $(S, \vec{k})$ is usually described as follows: C_0 is oriented counterclockwise, while $C_1, \ldots, C_m$ are oriented clockwise.

(2.4.4) Example. Let D be a two-dimension region

$$D := \left\{(x,y) \in \mathbb{R}^2 \,|\, (x-2)^2 + y^2 \geq 1,\ (x+2)^2 + y^2 \geq 1,\ 9x^2 + 16y^2 \leq 144\right\}.$$

The boundary ∂D of D is the disjoint union of two circles C, C' of radius 1 centered at $(2,0)$ and $(-2,0)$ respectively, and the ellipse

$$E := \{9x^2 + 16y^2 = 144\}.$$

[3] Some books may say something like "vectors $\vec{u}, \vec{v}, \vec{w}$ form a right-handed system if when you extend your right hand and make the three larger fingers orthogonal to each other, you can match $\vec{u}$ with the index finger, $\vec{v}$ with the middle and $\vec{w}$ with the thumb", plus a picture showing a right hand with the three fingers marked by $\vec{u}, \vec{v}$ and $\vec{w}$. But you can illustrate it equally well with your left hand: match $\vec{u}$ with the middle finger, $\vec{v}$ with the index finger and w with the thumb.

The standard convention is that a planar region in the (x,y)-plane is oriented by the constant vector field $\vec{k}$. The orientation of ∂D determined by the unit normal vector field $\vec{k}$ is the following:

- The outer part E of the boundary curve ∂D, which contains D in its interior, is oriented *counter-clockwise*.
- The circles C and C', which are contained in the interior of the ellipse E, are oriented *clockwise*.

(2.4.5) Example. Let T be the torus in $\mathbb{R}^3$ obtained by rotating the circle

$$C_1 = \{(x,0,z) \in \mathbb{R}^3 : (x-2)^2 + z^2 = 1\}$$

on the (x,z)-plane about the z-axis. Let vector $\vec{n}$ be the vector field on C_1 such that

$$\vec{n}(x,0,z) = (x-2)\vec{i} + z\vec{k} \qquad \text{for all } (x,0,z) \in C_1 .$$

Let S be the subset of T consisting of all point $(x,y,z) \in T$ with $x,y \geq 0$. Let $\vec{N}$ be the normal vector field on S obtained from $\vec{n}$ by rotating the latter vector field on C_1 about the z-axis. The boundary of the oriented surface $(S,\vec{N})$ is the disjoint union of two circles, the circle C_1 on the (x,z)-plane and the circle

$$C_2 = \{(0,y,z) \in \mathbb{R}^3 : (y-2)^2 + z^2 = 1\}$$

on the (y,z)-plane.

We want to pin down the orientation on $\partial S = C_1 \cup C_2$. One can achieve this by giving the values of unit tangent vectors $\vec{t}$ along tangent direction/orientation of ∂S at two points of ∂S, one in each component. We take the two points to be $P_1 = (3,0,0) \in C_1$ and $P_2 = (0,3,0) \in C_2$ respectively. The values of $\vec{N}$, $\vec{v}$ and $\vec{t}$ at P_1 and P_2 are worked out below.

- At P_1 we have $\vec{N}(P_1) = \vec{i}$, $\vec{v}(P_1) = -\vec{j}$. so $\vec{T}(P_1) = -\vec{k}$.
- At P_2 we have $\vec{N}(P_2) = \vec{j}$, $\vec{v}(P_2) = -\vec{i}$, so $\vec{T}(P_2) = \vec{k}$.

(2.5) Exercises.

(2.5.1) Let D be the solid

$$D := \{(x,y,z) \in \mathbb{R}^3 \,|\, x,y,z \geq 0,\ 0 \leq x+y+z \leq 3\}$$

The boundary ∂D of D is a closed surface, and is a union of 4 triangles. Let $\vec{N}$ be the unit normal vector field on ∂D pointing *away* from D.

(a) Give an explicit formula for the normal vector field $\vec{N}$.

(b) Let $S := \{(x,y,z) \in \partial D \,|\, x+y+z \geq 1\}$. The restriction of $\vec{N}$ to S is an orientation of S, which induces an orientation on the boundary ∂S of S. Describe explicitly the closed curve ∂S and the orientation of ∂S induced by the orientation $\vec{N}$ of S.

[Note that ∂S is a triangle.]

(2.5.2) Let C be the boundary of the surface

$$S = \{(x,0,z) \,|\, 1 \leq x^2 + 4z^2 \leq 4\}$$

on the (x,z)-plane. Note that C is a disjoint union of two ellipses on the (x,z)-plane. Orient S by the normal vector field $\vec{j}$. The orientation $\vec{N}$ of S defines an unit tangenet vector field $\vec{T}$, which gives C an orientation. Find $\vec{T}(1,0,0)$ and $\vec{T}(0,0,1)$.

(2.5.3) Let S be the surface

$$S := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 36,\ x - y - z + 10 \geq 0 \right\}$$

Let $\vec{N}$ be the continuous unit normal vector field on S such that $\vec{N}(6,0,0) = \vec{i}$. Let $C := \partial S$ be the boundary of S, and let $\vec{T}$ be the unit tangent vector field on C giving C the orientation induced by $(S,\vec{N})$. Compute $\vec{T}(4,8,6)$.
(Note that $(6,0,0)$ is a point of the surface S and $(4,8,6)$ is a point of the curve C.)

(2.5.4) Let S be the surface

$$S := \left\{ (y^2 + yz + z^2, y, z) \in \mathbb{R}^3 \;\middle|\; -10 \leq y, z \leq 10,\ (y-2)^2 + z^2 \geq 1,\ y^2 + (z-2)^2 \geq 0 \right\}.$$

You can think of S as the graph of the function

$$(y,z) \;\mapsto\; y^2 + yz + z^2$$

on the region of the (y,z)-plane consisting of all point lying inside a square edge length 20 and outside two circles of radius 1. Let $\vec{N}$ be the continuous unit vector field on S such that $\vec{N}(0,0,0) = -\vec{i}$. The boundary ∂S is the union of three piecewise smooth closed curves. Let $\vec{T}$ be the unit tangent vector field on the smooth locus of ∂S giving the orientation of ∂ induced by the oriented surface $(S,\vec{N})$. Compute $\vec{T}(100,10,0)$, $\vec{T}(9,3,0)$ and $\vec{T}(9,0,3)$.
(The three points $(100,10,0), (9,3,0)$ and $(9,0,3)$ lie on the three connected components of ∂S respectively.)

(2.5.5) Let D be the solid in $\mathbb{R}^3$ obtained by rotation the disk

$$\{(x,z) \,|\, (x-5)^2 + z^2 \leq 4\}$$

in the (x,z)-plane about the z-axis. Let ∂D be the boundary of D, which is a torus, oriented by the unit normal vector field $\vec{N}$ on ∂D pointing *away* from D.

(a) Let ∂D_- be the lower-half of ∂D, consisting of those points of ∂D whose z-coordinates are non-positive. Describe the projection of ∂D_- to the (x,y)-plane *explicitly*. (This is a region R in the (x,y)-plane.)

(b) The surface ∂D_- is the graph of a function $f : R \to \mathbb{R}$. Find this function f and write down the corresponding parametrization $\vec{r} = x\vec{i} + y\vec{j} + f\vec{k}$ of ∂D_-.

(c) Determine the sign function $(\partial D_-, \vec{N}, \vec{r})$ of for the parametrization $\vec{r}$ in (b) of the oriented surface $(\partial D_-, \vec{N})$.

(2.5.6) Let D be the solid in $\mathbb{R}^3$ obtained by rotation the disk

$$\{(x,z) \,|\, (x-5)^2 + z^2 \leq 4\}$$

in the (x,z)-plane about the z-axis. Let ∂D be the boundary of D, which is a torus, oriented by the unit normal vector field $\vec{N}$ on ∂D pointing *away* from D. Let S be the surface

$$\{(x,y,z) \in \partial D \,|\, 0 \leq x \leq 6\}.$$

Let $\vec{T}$ be the unit tangent vector field on the boundary ∂S of S induced by the oriented surface $(S,\vec{N})$. Note that ∂S is a disjoint union of two circles C_1, C_2 of radius 2 on the (y,z)-plane and a closed curve C_3 on the plane $\{x = 6\}$.

(a) Describe the two cicles C_1, C_2 and the unit tangent vector field $\vec{T}$ on them explicitly.

(b) Find $\vec{T}(6,0,\sqrt{3})$.

§3. Signed integrals

(3.1) Signed integrals in dimensions $n = 0, 3$ are easy to understand.

(a) $n = 0$. A 0-dimensional geometric body *with signs* has the form $\sum_{i=0}^{a} m_i \cdot [P_i]$, where each m_i is an integer, and each P_i is a point in $\mathbb{R}^3$. For a function $f(x,y,z)$,

$$\int_{\sum_i m_i \cdot [P_i]} f := \sum_i m_i \cdot f(P_i).$$

(b) $n = 3$. The convention is that we use the "standard orientation" for $\mathbb{R}^3$, and so for a solid B in $\mathbb{R}^3$ we just have the usual triple integral $\iiint_B f \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The sign (or orientation) of $\mathbb{R}^3$ is *concealed*—you don't see it in the notation because we have adopted the standard signed convention and assumed that everyone follows it.

(3.2) Line integrals: one-dimensional signed integrals
A 1-dimensional geometric body C with signs is a formal sum, of the form $C = \sum_{j=1}^{a} m_j \cdot (C_j, \vec{T}_j)$, where each m_i is an integer and each $(C_j, \vec{T}_j$ is an *oriented* curve. The integral over M of a one-dimensional integrand $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ has several equivalent notations,

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{x} = \int_C f_1\, dx + f_2\, dy + f_3\, dz.$$

The definition/meaning of the above line integral is as follows. If $\vec{r}_j(t)\colon [a_j, b_j] \to \mathbb{R}^3$ is a parametrization of the curve underlying a connected smooth *oriented* curve $C_j \subset \mathbb{R}^3$ for each $j = 1, \ldots, a$, then

$$\int_{C_j} \vec{F} \cdot \mathrm{d}\vec{x} = \mathrm{sign}(C_j, \vec{T}_j, \vec{r}_j) \cdot \int_{a_j}^{b_j} \left(\sum_{i=1}^{3} f_i(r_1(t), r_2(t), r_3(t)) \cdot \frac{dr_i(t)}{dt} \right) dt\,,$$

where $\mathrm{sign}(C_j, \vec{T}_j, \vec{r}_j)$ is 1 or -1 depending on whether the parametrization $\vec{r}_j$ is compatible with the orientation of C_i or not. [An oriented curve is a curve plus an *assigned* orientation.]

REMARK (a) If you use the notation $\vec{F} \cdot d\vec{x}$ or $f_1\, dx + f_2\, dy + f_3\, dz$, you simply substitute dx by $\frac{dx}{dt}\, dt$, dy by $\frac{dy}{dt}\, dt$ and dz by $\frac{dz}{dt}\, dt$ for each component C_j of C using the parametrization $\vec{r}_j(t)$ of C_j. Do not forget to examine whether the orientation of C_j is compatible with the parametrization $\vec{r}_j$ and figure out the sign $\mathrm{sign}(C_j, \vec{r}_j)$. Equivalently, make the following substitution

$$\vec{T}_j\, ds \rightsquigarrow \mathrm{sign}(C_j, \vec{T}_j, \vec{r}_j) \cdot \frac{d\vec{r}_j}{dt}\, dt$$

when you compute the oriented integral $\int_{C_j} \vec{F} \cdot \vec{T}_j\, ds$ using a parametrization $\vec{r}_j(t)$ of C_j.

(b) If you use the notation $\vec{F} \cdot \vec{T}\, ds$, please keep in mind that there is no need to compute $\vec{T}$ or ds. It is a waste of time if you do that. Just remember that $\vec{T}\, ds$ is a shorthand for $d\vec{x}$, and perform the substitution as in (a).

(3.3) Oriented surface integrals

A two-dimensional geometric body with signs is a formal sum of the form

$$S = \sum_{j=1}^{s} m_j \cdot (S_j, \vec{N}_j),$$

where each $(S_j, \vec{N}_j)$ is an oriented surface and each m_j is an integer. A general oriented surface integral of a vector field $\vec{F}$ over $\sum_{j=1}^{s} m_j \cdot (S_j, \vec{N}_j)$ has the form

$$\iint_{\sum_{j=1}^{s} m_j \cdot (S_j, \vec{N}_j)} \vec{F} \cdot \vec{N}\, d\sigma = \sum_{j=1}^{s} m_j \iint_{S_j} \vec{F} \cdot \vec{N}_j\, d\sigma.$$

The meaning of the terms $\iint_{S_j, \vec{N}_j} \vec{F} \cdot \vec{N}_j\, d\sigma$ is defined below.

(3.3.1) Definitioin. The integral

$$\iint_S \vec{F} \cdot \vec{N}\, d\sigma$$

for a vector field $\vec{F}$ with components f_1, f_2, f_3 over a smooth *oriented* surface $(S, \vec{N})$, when S is parametrized by a *bounded* region $D \subset \mathbb{R}^2$ through a smooth function

$$\vec{r}\colon D \to \mathbb{R}^3, \quad \vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + r_3(u,v)\,\vec{k}, \quad (u,v) \in D$$

is defined to be

$$\iint_S \vec{F} \cdot \vec{N}\, \mathrm{d}\sigma := \mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \iint_D \det \begin{bmatrix} f_1(\vec{r}(u,v)) & \partial x(u,v)/\partial u & \partial x(u,v)/\partial v \\ f_2(\vec{r}(u,v)) & \partial y(u,v)/\partial u & \partial y(u,v)/\partial v \\ f_3(\vec{r}(u,v)) & \partial z(u,v)/\partial u & \partial z(u,v)/\partial v \end{bmatrix} du\,dv \qquad (3)$$

where $\mathrm{sign}(S, \vec{N}, \vec{r})$ is the sign of the parametrization $\vec{r}$ of the oriented surface $(S, \vec{N})$, defined in 2.2.3.

(3.3.2) Remark This definition and the change of variable formula guarantees that the integral is independent of the choice of parametrization. In general an oriented surface $(S, \vec{N})$ can be decomposed into a finite union of pieces, each of which admits a parametrization.

(3.3.3) Substitution rule for $\vec{N}\, d\sigma$ in oriented surface integrals

The 3×3 determinant in equation (3) is equal to

$$\vec{F}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)$$

You get the whole integrand

$$\mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \det \begin{bmatrix} f_1(\vec{r}(u,v)) & \partial x(u,v)/\partial u & \partial x(u,v)/\partial v \\ f_2(\vec{r}(u,v)) & \partial y(u,v)/\partial u & \partial y(u,v)/\partial v \\ f_3(\vec{r}(u,v)) & \partial z(u,v)/\partial u & \partial z(u,v)/\partial v \end{bmatrix} du\,dv$$

from $\vec{F} \cdot \vec{N}\, d\sigma$ if you make the following subsitution

$$\vec{N}\, d\sigma \quad \rightsquigarrow \quad \mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\,dv$$

in equation (3).

(3.3.4) Another notation for oriented surface integrals
There is another notation for the oriented surface integral of a vector field $\vec{F}$ with components f_1, f_2, f_3, over an oriented surface $(S, \vec{N})$, the right hand side of the following equality

$$\iint_S \vec{F} \cdot N \, d\sigma \; =: \; \iint_{(S,\vec{N})} f_1 \, dy \wedge dz + f_2 \, dz \wedge dx + f_3 \, dx \wedge dy$$

Often $(S, \vec{N})$ is abbreviated as S, with the symbol S denoting an *oriented* surface. The symbol "$\wedge$" is suppressed in many books when it is understood that the integral is an oriented one. The substitute rules when given a parametrization $\vec{r}(u,v)$ with component functions $x(u,v), y(u,v), z(u,v)$ are

$$dy \wedge dz \;\; \rightsquigarrow \;\; \mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \det \begin{bmatrix} \partial y(u,v)/\partial u & \partial y(u,v)/\partial v \\ \partial z(u,v)/\partial u & \partial z(u,v)/\partial v \end{bmatrix} du \, dv$$

$$dz \wedge dx \;\; \rightsquigarrow \;\; \mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \det \begin{bmatrix} \partial z(u,v)/\partial u & \partial z(u,v)/\partial v \\ \partial x(u,v)/\partial u & \partial x(u,v)/\partial v \end{bmatrix} du \, dv$$

and

$$dx \wedge dy \;\; \rightsquigarrow \;\; \mathrm{sign}(S, \vec{N}, \vec{r}) \cdot \det \begin{bmatrix} \partial x(u,v)/\partial u & \partial x(u,v)/\partial v \\ \partial y(u,v)/\partial u & \partial y(u,v)/\partial v \end{bmatrix} du \, dv$$

Of course the whole effect is the cofactor expansion the 3×3 determinant in equation (3) along the first column of the same 3×3 matrix.

(3.3.5) Example. Suppose that S is the square $\{(x, -1, z) : -1 \leq x, z \leq 1\}$ on the plane $\{y = -1\}$, oriented by the unit normal vector field $\vec{N} = -\vec{j}$. We can choose

$$D = \{(u,v) \in \mathbb{R}^2 | -1 \leq u, v \leq 1\}, \quad \vec{r}(u,v) = (v, -1, u) \, .$$

Then $\partial r/\partial u = \vec{k}$, $\partial r/\partial v = \vec{i}$, $\mathrm{sign}(\vec{N}, \vec{r}) = -1$, and the oriented surface integral $\iint_S \vec{F} \cdot \vec{N} \, d\sigma$ becomes

$$-\iint_{-1 \leq u,v \leq 1} f_2(v, -1, u) \, du \, dv = -\iint_{-1 \leq x,z \leq 1} f_2(x, -1, z) \, dz \, dx \, .$$

§4. The fundamental theorem of calculus

(4.1) The fundamental theorem of calculus in higher dimensions has the general form

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha \, ,$$

where

- α is a n-dimensional integrand, usually specified by a number of scalar-valued funtions, and
- M is an $n+1$-dimensional *priented* geometric body M with signs specializes to various forms in different dimensions.

We will explain the cases when $M \subset \mathbb{R}^3$ and $n = 0, 1$ or 2. The case $n = 0$ is not too different from the usual fundamental theorem of calculus, while the restriction $M \subset \mathbb{R}^3$ makes the case $n = 2$ easier to explain. The case when $n = 1$ is the Stokes' theorem, and needs to make the orientations for M and its boundary ∂M *compatible* so that the equality holds.

(4.2) The Cases when $n=0$ and $n=2$

A. $n=0$. This is the fundamental theorem of calculus for line integrals of *conservative* vector fields:

For any smooth function f defined on an open subset of $\mathbb{R}^3$ containing the given oriented curve C, we have

$$\int_{\partial C} f = \int_C \operatorname{grad}(f)\cdot d\vec{x}.$$

The boundary ∂C of C is empty if C is a closed curve, in which case the above equality reduces to $\oint_C \operatorname{grad}(f)\cdot d\vec{x}=0$. If C is connected and goes from P to Q according to its orientation, the $\partial C = [Q]-[P]$.

B. $n=2$ (**Gauss/divergence theorem**): For any bounded solid B in $\mathbb{R}^3$, orient its boundary ∂B by the unit normal vector field $\vec{N}$ on ∂B which points *away* from B. Notice that the boundary ∂B of B may not be connected. On each connected component of ∂B we have a orientation given by this recipe. pointing *away* from B.

With this choice of orientation of the boundary ∂B and for every vector field $\vec{F}$ *smooth on B*, we have

$$\iint_{\partial B} \vec{F}\cdot\vec{N}\,d\sigma = \iiint_B \operatorname{div}(F)\,dx\,dy\,dz.$$

Consequence: Suppose that a closed oriented surface $(S_1,\vec{N}_1)$ can be smoothly deformed to a closed oriented surface $(S_2,\vec{N}_2)$ so that $(S_1,\vec{N}_1)\cup(S_2,-\vec{N}_2)$ is the boundary of solid $B\subset\mathbb{R}^3$. Suppose that $\vec{F}$ is smooth on B (i.e. the deformation from S_1 to S_2 through the solid B does not meet any singularity of $\vec{F}$), then

$$\iint_{S_1} \vec{F}\cdot\vec{N}_1\,d\sigma = \iint_{S_2} \vec{F}\cdot\vec{N}_2\,d\sigma \quad \text{if } \operatorname{div}(\vec{F})=0.$$

This statement includes the extreme case when S_2 is empty and $(S_1,\vec{N}_1)$ is the oriented boundary of B:

$$\iint_S \vec{F}\cdot\vec{N}\,d\sigma = 0 \quad \text{if } \operatorname{div}(\vec{F})=0,\ S=\partial B \text{ and } \vec{F} \text{ is smooth on } B.$$

(4.3) Stokes' theorem: the case when $n=1$

(4.3.1) The set-up is as follows. We are given an oriented surface $(S,\vec{N})$ in $\mathbb{R}^3$, which may or may not be closed.[4] We orient the boundary curve ∂S of S by the unit tangent vector field $\vec{T}$ on ∂S as in 2.4. Stokes' theorem asserts that

$$\int_{\partial S} \vec{F}\cdot\vec{T}\,ds = \iint_S \operatorname{curl}(F)\cdot\vec{N}\,\mathrm{d}\sigma$$

holds for every vector field $\vec{F}$ smooth on an open subset of $\mathbb{R}^3$ containing S, i.e. none of the component functions f_i of $\vec{F}$ has any singularity. on the surface S. When S is a closed surface, its boundary ∂S is empty, and Stokes' theorem simply says that

$$\oiint_S \operatorname{curl}(F)\cdot\vec{N}\,d\sigma = 0$$

for closed surfaces.

[4]The boundary ∂S of S is empty if S is a closed surface, but in general ∂S may consists of several connected closed curves in $\mathbb{R}^3$.

We repeat an important point: the line integral on the left hand side of the above formula is computed according to the orientation of the boundary ∂S given by the unit tangent vector field $\vec{T}$ on ∂S determined by the orientation $\vec{N}$ of S. The surface integral and the line integral in the statement of Stokes' theorem are both signed integrals. The signs produces necessary cancellations so that the theorem holds.

(4.3.2) Applications of Stokes' theorem.

(a) Suppose that S_1 and S_2 are two smooth surfaces in $\mathbb{R}^3$ oriented by unit normal vector fields N_1 and N_2 respectively. Let $\vec{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ which contains both S_1 and S_2. Suppose moreover that $\partial S_1 = \partial S_2$ as oriented closed curves. If there exists a smooth vector field $\vec{G}$ such that $\vec{F} = \operatorname{curl}(\vec{G})$, then

$$\iint_{S_1} \vec{F} \cdot \vec{N}_1 \, d\sigma = \iint_{S_1} \vec{F} \cdot \vec{N}_1 \, d\sigma$$

REMARK This is a generalization of the statement that the line integrals of a conservative vector field is independent of path.

(b) Suppose that an oriented curve C_1 is deformed to an oriented curve C_2 in such a way that $C_1 - C_2$ is the *oriented* boundary of an oriented surface $(S, \vec{N})$ in $\mathbb{R}^3$. Suppose moreover that $\operatorname{curl}(\vec{F}) = 0$ and $\vec{F}$ is smooth on S; in other words the deformation of C_1 to C_2 through the surface S does not meet any singular point of $\vec{F}$. Then

$$\int_{C_1} \vec{F} \cdot d\vec{x} = \int_{C_2} \vec{F} \cdot d\vec{x},$$

under the assumptions that (i) $\operatorname{curl}(\vec{F}) = 0$ and (ii) $C_1 - C_2$ is the oriented boundary of a smooth surface S such that $\vec{F}$ is smooth on S (i.e. no singularity of $\vec{F}$ lies on S).

REMARK (1) This statement includes the extreme case when $C_2 = \emptyset$ and $C_1 = \partial S$.

(2) Note that it is not assumed that there exists a smooth function f such that $\operatorname{grad}(f) = \vec{F}$.

§5. Line and surface integrals without signs

There is not much to say about line integrals and surface integrals without signs. The fundamental theorem of calculus holds only for signed integrals; this is already the case for integrals in one variable. We will recall their definitions below. You can still use the fundamental theorem of calculus, but you need to convert an integral without signs to a signed one before applying that theorem. These integrals can be evaluated using any parametrization, so choose a good parametrization when you do the computation.

(5.1) Unsigned line integrals

Let C be a curve parametrized by a smooth function $\vec{r} : I \to \mathbb{R}^3$ on a finite interval $I \subset \mathbb{R}$, and let $f(x,y,z)$ be a continuous function on C. The unsigned line integral is

$$\int_C f(x,y,z) \, ds := \int_I f(\vec{r}(t)) \left\| \frac{d\vec{r}}{dt} \right\| dt.$$

In other words, substitute the line element ds for the curve C by

$$\left\| \frac{d\vec{r}}{dt} \right\| dt$$

when you use a parametrization $\vec{r}(t)$ to evaluate an unsigned line integral $\int_C f(x,y,z) \, ds$.

(5.2) Unsigned surface integrals

Suppose that S is a smooth surface in $\mathbb{R}^3$ parametrized by a smooth function $\vec{r} : D \to \mathbb{R}^3$ on a bounded region $D \subset \mathbb{R}^2$. Let f be a continuous function on S. The unsigned surface integral is

$$\iint_S f(x,y,z)\,d\sigma := \iint_D f(\vec{r}(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

In other words, substitute the area element $d\sigma$ by

$$\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

you use a parametrization $\vec{r}(u,v)$ to evaluate an unsigned surface integral $\iint_S f(x,y,z)\,d\sigma$.