Math 240 Practice Problems, February 2015

1. (a) Find all solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \\ 6 \end{bmatrix}$$

such that $x_3 \cdot x_4 = 0$.

(b) Compute the determinant of the square matrix in (a) above.

2. Six of the following 7 elements of $\mathbb{R}^3$,

$$(1,1,-4),\ (3,5,-26),\ (2,1,1),\ (5,6,-27),\ (4,1,5),\ (5,-2,29),\ (6,2,4)$$

lie on a 2-dimensional subspace (i.e. a plane passing through the origin). Which one does not lie on the plane that contains the other six points?

3. Compute the determinants of the following matrices:

(a) $$\begin{pmatrix} & & & & & & & 1 \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ 1 & & & & & & & \end{pmatrix}$$

(b) $$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

4. Let B be the square matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$$

Then the trace of B^{-1} is equal to

A. 21

B. 13

C. -15

D. 0

E. -14

F. None of the above.

5. Consider the following system of linear equations in 6 variables

$$\begin{bmatrix} 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) What is the rank of the above 3×6 matrix?

(b) Write down the general solution of the above system of equations, in the form of generic linear combination of a number of linearly independent solutions.

6. Let A be the 3×5 matrix $A = \begin{pmatrix} 3 & 2 & 1 & -1 & -2 \\ 1 & 1 & 2 & 1 & 3 \\ 3 & 1 & -4 & -5 & -13 \end{pmatrix}$.

(a) Determine the dimension of the linear span of the columns of A.

(b) For which values of the parameter c will the system of equations

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ c \end{pmatrix}$$

admit at least a solution?

7. Provide an example in each of the following questions.

(a) Two square matrices A, B of the same size such that $A \cdot B \neq B \cdot A$.

(b) A non-zero square matrix C such that $C^2 = 0$

(c) Exhibit three different bases of $\mathbb{R}^3$.

(d) A *non-linear* map from the real vector space of all real-valued continuous functions on $\mathbb{R}$ to itself.

(e) Two 3×3 matrices A, B such that $\det(A + B) \neq \det(A) + \det(B)$.

8. Let Q be the vector space consisting of all polynomials in two variables x and y of total degree at most 3. Let $T : Q \to Q$ be the linear transformation with sends every element $f(x, y) \in Q$ to

$$(2x - y)\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

(a) Find a basis B of Q.

(b) Determine the matrix $[T]_B^B$ for the basis B you gave in (a).

(c) Compute the determinant of the matrix $[T]_B^B$.

(d) Find a basis of the kernel $\mathrm{Ker}(T)$ and determine the dimension of $\mathrm{Ker}(T)$.

(e) What is the dimension of the vector subspace W of Q consisting of all polynomials $g(x, y)$ which can be expressed as $g = T(f)$ for some element f in Q?

9. True/False questions.

(a) The set $\mathbb{R}[x]$ of all polynomials in x with real coefficients form a vector space, where vector addition is sum of polynomials and scalar multiplication is multiplying a polynomial by a real number. Let S be the subset of $\mathbb{R}[x]$ consisting of all polynomials $f(x)$ such that $\int_0^1 f(x)\,dx = 1$. This subset S is a vector subspace of $\mathbb{R}[x]$.

(b) Let P_{10} (respectively P_{12}) be the vector space of all polynomials in one variable x with coefficients in $\mathbb{R}$ of degree at most 10 (respectively). Let $T : P_{10} \to P_{12}$ be a linear transformation. There exists an element $f(x) \in P_{12}$ such that $f \neq T(g)$ for every element $g(x) \in P_{10}$.

(c) If A is a 6×6 matrix and $A = -A^T$, then $\det(A) = 0$.

(d) If B is a 5×5 matrix and $A = A^T$, then $\det(B) = 0$.

(e) Suppose $\vec{a}_1, \ldots, \vec{a}_9$ are 9 elements of $\mathbb{R}^7$. There exist a 9×9 matrix A with $\mathrm{rank}(A) = 3$ such that $A \cdot \vec{a}_i = \vec{0}$ for all $i = 1, \ldots, 9$.

(f) Let A be a 5×5 matrix. Then $\det(2 \cdot A) = 2 \cdot \det(A)$.

(g) Let A be a 5×5 matrix. If the sum of the 5 columns of A is the zero vector, then $\det(A) = 0$.

(h) Suppose A is a 7×6 matrix and B is a 6×7 matrix. If $B \cdot A$ is an invertible 6×6 matrix, then the dimension of the null space of A is 1.