Math 240 Practice Problems Set 2, March 2015

1. Which ones of the following limits exits? Explain your reasons for each of the limits.

(a) $\lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n$

(b) $\lim_{n\to\infty} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^n$

(c) $\lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$

(d) $\lim_{n\to\infty} \begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}^n$

(e) $\lim_{n\to\infty} \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}^n$

2. Find a formula for $\lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n$ valid for every positive integer n.

3. Let A be the 4×4 matrix

$$A = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A.

4. Let B be the 4×4 matrix

$$B = \frac{1}{2} \cdot \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Does there exist an invertible 4×4 matrix C such that $C^{-1} \cdot B \cdot C$ is diagonal? Find such a matrix C if there is one, or explain why such a matrix C does not exist.

5. (a) Find the 2×2 matrix A such that $\vec{x} \mapsto A \cdot \vec{x}$ for $\vec{x} \in \mathbb{R}^2$ is the counter-clockwise rotation about the origin by 45°.

(b) Does there exist an invertible 2×2 matrix C with real entries such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix? Find such a matrix C if there is one, or explain why such a matrix C does not exist.

(c) Does there exist an invertible 2×2 matrix C with complex entries such that $D^{-1} \cdot A \cdot D$ is a diagonal matrix? Find such a matrix D if there is one, or explain why such a matrix D does not exist.

6. Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 9 \end{pmatrix}$. Compute e^A explicitly.

7. Find the general solution of the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 1 = e^{-x} + \cos x - 1$$

8. Find the general solution of the differential equation

$$\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 5\right)^2 y = e^{(-1+2\sqrt{-1})x}$$

9. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$$

on the half-line $x > 0$.

10. Determine all solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = \cos(3x)$$

such that $\lim_{x\to\infty} y(x) = 0$, or explain why no such solution exists.

11. Is there a solution $y(x)$ of the differential equation

$$\left(\frac{d^2y}{dx^2} + 4\right)^2 y = \sin(2x)$$

such that $y(x)$ is *bounded* on $\mathbb{R}$ (in the sense that there exists a constant $C > 0$ such that $|y(x)| \leq C$ for all $x \in \mathbb{R}$)? Find all bounded solutions if they exist, and explain why every solution is unbounded.