MATH 240 ASSIGNMENT 4, SPRING 2015

Note: There are only five plus two extra problems in this assignment because of the first midterm exam on Tuesday February 17. There will be no HW due on Friday February 20. (But please do the practice problems for the first midterm.)

Due in class on Friday, February 13

Part 1.

(a) Read DELA §4.7, §4.8, §5.5 and the following recap:

a1. Suppose $B = (v_1, \ldots, v_n)$ is a basis of an F-vector space V. This basis B provides a linear coordinate system for V: for a vector $x \in V$, its coordinate $[x]_B$ is the *column* vector in F^n such that $(v_1, \ldots, v_n) \cdot [x]_B = x$, i.e.

$$x = B \cdot [x]_B \qquad \text{for every vector } x \in V \tag{1}$$

a2. Suppose we have another basis $D = (u_1, \ldots, u_n)$ of V. The change of basis matrix $P_{C \leftarrow D}$ is the $n \times n$ matrix satisfying

$$B = D \cdot P_{D \leftarrow B} \tag{2}$$

The following equality holds for every vector $x \in V$:

$$D \cdot [x]_D = x = B \cdot [x]_B = D \cdot P_{D \leftarrow B} \cdot [x]_B.$$

(The first two equality comes from the definition of the coordinates systems attached to the two bases D and B respectively, while the third equality comes from the definition of $P_{D \leftarrow B}$.) The above displayed formula shows that

$$[x]_D = P_{D \leftarrow B} \cdot [x]_B \qquad \text{for every vector } x \in V, \tag{3}$$

a relation between the two linear coordinate systems on V.

a3. Suppose that $T : V \to W$ is a linear transformation between two vector spaces over the same field F of scalars. Suppose moreover that $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_m)$ are basis for V and W respectively. The matrix $[T]_B^C$ of T for the bases B and C is

$$[T]_B^C := [T \cdot B]_C, \quad \text{equivalently} \quad C \cdot [T]_B^C = T \cdot B \tag{4}$$

where $T \cdot B$ stands for $(Tv_1, \ldots, Tv_n)$. For any $x \in V$, we have $x = B \cdot [x]_B$ and

$$C \cdot [Tx]_C = T(x) = T(B[x]_B) = T \cdot B \cdot [x]_B = C \cdot [T]_B^C \cdot [x]_B,$$

so

$$[Tx]_C = [T]_B^C \cdot [x]_B \qquad \text{for all } x \in V. \tag{5}$$

(b) Do (but do not hand in) the following problems from DELA:

§4.7 T/F Review 5, 6, 7; Problems 35, 38

§4.8 Problems 3, 4, 5

§4.9 Problems 6, 11, 18

§5.1 Problems 4, 5, 6, 23

Part 2. Do and write up the following problems from DELA:

§4.7 Problem 24

§4.9 Problem 16

§5.1 Problem 30

§5.5 Problems 14, 20

Part 3. Extra credit problems:

(i) Let V be the real vector space consisting of all polynomials in 3 variables x, y, z of total degree at most 2. (For instance the total degree of the polynomial $x^3 + xyz + y^2z^2 + xy - 3yz + 3$ is 4, because the monomial y^2z^2 has degree 4, while the other terms all have degree at most 3.) Vector addition in V are sum of polynomials and scalar multiplication is multiplying a polynomial with real numbers.

(a) Is the map $\ell : V \to \mathbb{R}^2$ which sends every polynomial $f(x, y, z) \in V$ to the real number

$$\int_2^5 f(t^2, t, -t)\, dt$$

an $\mathbb{R}$-linear transformation?

(b) Show that $\dim_{\mathbb{R}}(V) = 10$.

(c) Let W be the real vector space consisting of all triples $(g_1(x, y, z), g_2(x, y, z), g_3(x, y, z)$ where g_1, g_2, g_3 are polynomials in x, y, z of degree at most 2. What is the dimension of W?

(d) Let $T : V \to W$ be the linear transformation which sends every polynomial $f(x, y, z) \in V$ to the element $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \in W$. Let B, C be bases of V and W respectively consisting of polynomials in V and W. Write down the matrix for T for the bases B and C.

(ii) Let P_{99} be the 100-dimensional vector space over $\mathbb{R}$ consisting of all polynomials in one variable x of degree at most 99 with coefficients in $\mathbb{R}$. Let $S : P_{99} \to \mathbb{R}^{100}$ be the linear transformation defined by

$$S : f(x) \mapsto (f(0), f(1), f(2), \ldots, f(99)) \qquad \text{for every } f(x) \in P_{99}.$$

(a) Determine the kernel $\mathrm{Ker}(S)$ of S

(b) Show that S has an inverse.