Part 1.

(a) Read DELA §§6.1–6.3

Note about the annihilator of a function $f(x)$ in a linear differential equation with constant coefficients $P\left(\frac{d}{dx}\right)y = f(x)$, discussed in §6.3: This notion of annihilator work well when the coefficients of the linear differential equation are all constants, and when the function $f(x)$ is of the form

$$f(x) = g_1(x)e^{a_1 x} + g_2(x)e^{a_2 x} + \cdots + g_k(x)e^{a_k x},$$

where $g_1(x), \ldots, g_m(x)$ are polynomials with coefficients in $\mathbb{C}$ and $a_1, \ldots, a_m$ are constants in $\mathbb{C}$. In a way the point of the discussions about the notion of annihilators is to justify the following choice of the general shape of possible “particular solution” $y_p(x)$ of the equation $P\left(\frac{d}{dx}\right)y = f(x)$:

$$y_p(x) = x^{m_1}h_1(x)e^{a_1 x} + \cdots + x^{m_k}h_k(x)e^{a_k x},$$

where $m_i$ is the multiplicity of $a_i$ as a possible root of the polynomial $P(\lambda)$, and $h_i(x)$ is a polynomial of degree equal to the polynomial $g_i(x)$, for each $i = 1, \ldots, k$. Note that $m_i = 0$ if $P(a_i) \neq 0$. The coefficients of the polynomials are to be determined so that $y_p(x)$ satisfies $P\left(\frac{d}{dx}\right)y_p = 0$; they are the “undetermined coefficients”. The task of finding them boils down to solving a system of linear equations in these undetermined coefficients.

(b) Do (but do not hand in) the following problems from DELA:

§6.1 T/F Review 1, 3, 4, 5, 7; Problems 31, 32, 36, 39
§6.2 T/F Review 1, 2, 8; Problems 21, 25, 37, 38, 39
§6.3 T/F Review 2, 6, 7, 8; Problems 26, 31, 36, 40

Part 2. Do and write up the following problems from DELA:

§6.1 Problems 28, 30, 38
§6.2 Problems 28, 34, 40
§6.3 Problems 28, 32, 38

Part 3. Extra credit problems.

E1. A linear differential operator of order $n$ in one variable $x$ is an expression of the form

$$P\left(x, \frac{d}{dx}\right) = a_n(x) \frac{d^n}{dx^n} + a_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x) \frac{d}{dx} + a_0(x),$$

where $a_n(x), \ldots, a_1(x), a_0(x)$ are (nice) functions in $x$, and $a_n(x)$ is not identically zero. A linear differential operator $P(x, \frac{d}{dx})$ as above sends any function $f(x)$ to the function

$$\left(P\left(x, \frac{d}{dx}\right)f\right)(x) := a_n(x) \frac{d^n f}{dx^n} + a_{n-1}(x) \frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_1(x) \frac{d f}{dx} + a_0(x) f(x).$$
(a) Let \( g(x) \) be a (nice) function in \( x \). Then (multiplication by) \( g(x) \) is a linear differential operator of order 0, while \( \frac{d}{dx} \) is a first order linear differential operator. Show that the composition \( \frac{d}{dx} \circ g(x) \), which sends every function \( f(x) \) to the function \( \frac{d}{dx} (g(x) \cdot f(x)) \), is a first order linear differential operator.

Note: The composition \( g(x) \circ \frac{d}{dx} \) is a first order linear differential operator by definition. The content of (a) is to write the composition \( \frac{d}{dx} \circ g(x) \) in the standard form \( a_1(x) \frac{d}{dx} + a_0(x) \) for suitable functions \( a_1(x) \) and \( a_0(x) \).

(b) Suppose that \( P(\langle x, \frac{d}{dx} \rangle) \) and \( Q(\langle x, \frac{d}{dx} \rangle) \) are linear differential operators of order \( n \) and \( m \) respectively. Show that the composition \( P(\langle x, \frac{d}{dx} \rangle) \circ Q(\langle x, \frac{d}{dx} \rangle) \) is a linear differential operator of order at most \( n + m \).

(c) Show that for every positive integer, the \( n \)-th iterate \( (x \frac{d}{dx})^n \) of the differential operator \( x \frac{d}{dx} \) is a linear combination of differential operators \( x^j \frac{d^j}{dx^j} \) for \( j = 0, \ldots, n \).

(d) Show that \( \frac{d}{dx} \circ x - x \circ \frac{d}{dx} = 1 \).

E2. Continue with the notation in E1.

(a) Find a formula which expresses the composition \( \frac{d^n}{dx^n} \circ g(x) \) as a linear differential operator of order at most \( n \).

(b) Let us indulge in the freedom of expression and agree to consider all infinite series of the form

\[
P(x, D) = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x) + a_{-1}(x)D^{-1} + a_{-2}(x)D^{-2} + \cdots,
\]

where \( D := \frac{d}{dx} \) and the sum runs through all integers \( \leq n \), and \( n \) is allowed to be a negative integer. (Try to consider the above infinite series formally and not to worry about its effect when applied to a function \( f(x) \). You can imagine \( D^{-1}, D^{-2}, \ldots \) as some sort of “integral operators”. Here we consider theses expressions only formally and try to construct some meaningful algebraic operations.) Extrapolate the formula you found in E2(a) and give a “good” definition of composition of two such “extended differential operators” \( P \) and \( Q \). If your definition is indeed a good one, the composition would satisfy the associativity rule. Moreover it would extend the definition for linear differential operators as in E1.