MATH 240 ASSIGNMENT 6, SPRING 2015

Due in class on Friday, March 6

Part 1.

(a) Read DELA §§6.1–6.3

Note bout the *annihilator* of a function $f(x)$ in a linear differential equation with *constant coefficients* $P\left(\frac{d}{dx}\right)y = f(x)$, discussed in §6.3: This notion of annihilator work well when the coefficients of the linear differential equation are all *constants*, and when the function $f(x)$ is of the form

$$f(x) = g_1(x)e^{a_1 x} + g_2(x)e^{a_2 x} + \cdots + g_k(x)e^{a_k x},$$

where $g_1(x), \ldots, g_m(x)$ are polynomials with coefficients in $\mathbb{C}$ and $a_1, \ldots, a_m$ are constants in $\mathbb{C}$. In a way the point of the discussions about the notion of annihilators is to justify the following choice of the general shape of possible "particular solution" $y_{\mathrm{p}}(x)$ of the equation $P\left(\frac{d}{dx}\right)y = f(x)$:

$$y_{\mathrm{p}}(x) = x^{m_1} h_1(x)e^{a_1 x} + \cdots + x^{m_k} h_k(x)e^{a_k x},$$

where m_i is the multiplicity of a_i as a possible root of the polynomial $P(\lambda)$, and $h_i(x)$ is a polynomial of degree equal to the polynomial $g_i(x)$, for each $i = 1, \ldots, k$. Note that $m_i = 0$ if $P(a_i) \neq 0$. The coefficients of the polynomials are to be determined so that $y_{\mathrm{p}}(x)$ satisfies $P\left(\frac{d}{dx}\right)y_{\mathrm{p}} = 0$; they are the "undetermined coefficients". The task of finding them boils down to solving a system of linear equations in these undetermined coefficients.

(b) Do (but do not hand in) the following problems from DELA:

§6.1 T/F Review 1, 3, 4, 5, 7 ; Problems 31, 32, 36, 39

§6.2 T/F Review 1, 2, 8; Problems 21, 25, 37. 38, 39

§6.3 T/F Review 2, 6, 7, 8; Problems 26, 31, 36, 40

Part 2. Do and write up the following problems from DELA:

§6.1 Problems 28, 30, 38

§6.2 Problems 28, 34, 40

§6.3 Problems 28, 32, 38

Part 3. Extra credit problems.

E1. A linear differential operator of order n in one variable x is an expression of the form

$$P\left(x, \frac{d}{dx}\right) = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)\,,$$

where $a_n(x), \ldots, a_1(x), a_0(x)$ are (nice) functions in x, and $a_n(x)$ is not identically zero. A linear differential operator $P(x, \frac{d}{dx})$ as above sends any function $f(x)$ to the function

$$\left(P\left(x, \frac{d}{dx}\right)f\right)(x) := a_n(x)\frac{d^n f}{dx^n} + a_{n-1}(x)\frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_1(x)\frac{df}{dx} + a_0(x)\,f(x).$$

(a) Let $g(x)$ be a (nice) function in x. Then (multiplication by) $g(x)$ is a linear differential operator of order 0, while $\frac{d}{dx}$ is a first order linear differential operator. Show that the composition $\frac{d}{dx} \circ g(x)$, which sends every function $f(x)$ to the function $\frac{d}{dx}(g(x) \cdot f(x))$, is a first order linear differential operator.

Note: The composition $g(x) \circ \frac{d}{dx}$ is a first order linear differential operator by definition. The content of (a) is to write the composition $\frac{d}{dx} \circ g(x)$ in the standard form $a_1(x)\frac{d}{dx} + a_0(x)$ for suitable functions $a_1(x)$ and $a_0(x)$.

(b) Suppose that $P(\big(x, \frac{d}{dx}\big)$ and $Q(\big(x, \frac{d}{dx}\big)$ are linear differential operators of order n and m respectively. Show that the composition $P(\big(x, \frac{d}{dx}\big) \circ Q(\big(x, \frac{d}{dx}\big)$ is a linear differential operator of order at most $n+m$.

(c) Show that for every positive integer, the n-th iterate $\big(x\frac{d}{dx}\big)^n$ of the differential operator $x\frac{d}{dx}$ is a linear combination of differential operators $x^j \frac{d^j}{dx^j}$ for $j = 0, \ldots, n$.

(d) Show that $\frac{d}{dx} \circ x - x \circ \frac{d}{dx} = 1$.

E2. Continue with the notation in E1.

(a) Find a formula which expresses the composition $\frac{d^n}{dx^n} \circ g(x)$ as a linear differential operator of order at most n.

(b) Let us indulge in the freedom of expression and agree to consider all infinite series of the form

$$P(x,D) = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x) + a_{-1}(x)D^{-1} + a_{-2}(x)D^{-2} + \cdots,$$

where $D := \frac{d}{dx}$ and the sum runs through all integers $\leq n$, and n is allowed to be a negative integer. (Try to consider the above infinite series *formally* and not to worry about its effect when applied to a function $f(x)$. You can imagine $D^{-1}, D^{-2}, \ldots$ as some sort of "integral operators". Here we consider theses expressions only *formally* and try to construct some meaningful algebraic operations.) Extrapolate the formula you found in E2 (a) and give a "good" definition of composition of two such "extended differential operators" P and Q. If your definition is indeed a good one, the composition would satisfy the associativity rule. Moreover it would extend the defintion for linear differential operators as in E1.