MATH 240 ASSIGNMENT 8, SPRING 2015

Due in class on Friday, April 3, 2015

Included in this and the next assignment is a synopsis of the key idea behind Jordan forms, stated in a form suitable for application to system ofs first order linear ODE's with constant coefficients. Problems B1–B3 and C1 revisit previous materials in ODE and tie it with Jordan forms; they can be viewed as special cases treated in §7.5 of DELA. Problem C2 is about an important general property of the Wronskian; we have discussed the statement in C2(a) in class, and indicated why C2(b) follows from C2(a).

Theorem. (Jordan forms stated intrinsically. Let V be a finite dimensional vector space over $\mathbb{C}$, and let T be a linear transformation from V to itself. There exists a natural number $k \geq 1$, k complex numbers $\lambda_1, \ldots, \lambda_k$, not necessarily distinct, and a $\mathbb{C}$-basis

$$\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,d_1}; \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{1,d_2}; \ldots; \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{1,d_k}$$

of V, grouped in k cyclic blocks such that

$$\begin{aligned}(T-\lambda_i)(\mathbf{v}_{i,1}) &= 0\\ (T-\lambda_i)(\mathbf{v}_{i,2}) &= \mathbf{v}_{i,1}\\ (T-\lambda_i)(\mathbf{v}_{i,3}) &= \mathbf{v}_{i,2}\\ &\cdots\\ (T-\lambda_i)(\mathbf{v}_{i,d_i}) &= \mathbf{v}_{i,d_{i-1}}\end{aligned} \tag{1}$$

Remark. (a) The characteristic polynomial of T is

$$\pm\prod_{i=1}^{k}(\lambda-\lambda_i)^{d_i} = \pm(\lambda-\lambda_1)^{d_1}\cdot(\lambda-\lambda_k)^{d_k}$$

(b) T is diagonalizable if and only if $d_1 = d_2 = \cdots = d_k = 1$. In other words T is defective if and only if $d_i > 1$ for at least one i, for some integer i between 1 and k.

Part A. (a) Read the above summary for Jordan forms, and also §§7.1–7.4.

(b) Do and write up the answer to the following problems in DELA.

§7.1 Problem 20.

§7.2 Problem 12.

§7.3 Problem 4.

§7.4 Problems 12, 18, 24, 26.

Part B. Let V be the vector space over $\mathbb{C}$ consisting of all solutions $y(x)$ of a differential equation

$$\left(\frac{d}{dx}-a_1\right)^{e_1}\cdots\left(\frac{d}{dx}-a_r\right)^{e_r} y(x) = 0 \tag{2}$$

with r mutually distinct complex numbers $a_1, \ldots, d_r$.

B1. Show that the differential operator $\frac{d}{dx}$ induces a linear transformation T from V to V itself.

[In other words, you need to show that the derivative $\frac{dy}{dx}$ of a solution $y(x)$ of (2) is again of solution of (2).]

B2. Find a positive integer k, k complex numbers $\lambda_1, \ldots, \lambda_k$ and $\mathbb{C}$-basis of V of V grouped in k blocks satisfying the condition (1) for Jordan forms.

B3. What is the matrix A for the linear transformation T with respect to the basis you found in question B2 above?

[The matrix A you get should be an $n \times n$ matrix with $n = d_1 + \cdots + d_r$.]

Part C. Extra credit problems.

C1. Let A be the $n \times n$ matrix you obtained in question B3, and let B be the $n \times n$ matrix

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & & & c_1 \\ 0 & 1 & 0 & \cdots & c_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & c_{n-1} \end{bmatrix}$$

where $c_0, c_1, \ldots, c_{n-1}$ are complex numbers such that

$$\left(\frac{d}{dx} - a_1\right)^{e_1} \cdots \left(\frac{d}{dx} - a_r\right)^{e_r} = \frac{d^n}{dx^n} - c_{n-1}\frac{d^{n-1}}{dx^{n-1}} - c_{n-2}\frac{d^{n-2}}{dx^{n-2}} - \cdots - c_1\frac{d}{dx} - c_0.$$

The (possibly) non-entries of B are: 1's immediately below the diagonal, and $c_0, \ldots, c_{n-1}$ in the last column.

(a) Use the method in §7.2 to show that ODE (2) is equivalent to the system of ODE 3

$$\frac{d}{dx}\mathbf{z} = B \cdot \mathbf{z}, \tag{3}$$

(b) Explain the reason why the linear system of ODE

$$\frac{d}{dx}\mathbf{y} = A \cdot \mathbf{y} \tag{4}$$

is equivalent to the ODE (2) in part B by showing that the matrices A and B are conjugate, i.e. there exists an invertible matrix C such that $A = C^{-1}B \cdot C$.

[Here $\mathbf{y}$ is an $n \times 1$ column vector representing an "unknown" function in the variable x. If the question looks too general at first, you might want to begin with the simpler special case when $r = 1$ in the notation of part B.]

C2. Let A be an $n \times n$ matrix.

(a) Let $\mathbf{v}_1, \ldots, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be n column vectors with n entries. Let C be the $n \times n$ matrix with $\mathbf{v}_1, \ldots, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as its n columns, in this order. For each $i = 1, \ldots n$, let B_i be the $n \times n$ matrix obtained by replacing the i-th column of C by the vector $A \cdot v_i$. Show that

$$\det(B_1) + \det(B_2) + \cdots + \det(B_n) = \mathrm{Tr}(A) \cdot \det(B).$$

(b) Use (a) to show that the Wronskian $W(\mathbf{x}_1, \cdots, \mathbf{x}_n)(t)$ of any n solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ of the system of linear differential equation

$$\frac{d}{dt}\mathbf{x}(t) = A(t) \cdot \mathbf{x}(t) \tag{5}$$

satisfies the differential equation

$$\frac{d}{dt}W(\mathbf{x}_1, \cdots, \mathbf{x}_n)(t) = \mathrm{Tr}(A(t)) \cdot W(\mathbf{x}_1, \cdots, \mathbf{x}_n). \tag{6}$$

Here $A(t)$ is a $n \times n$ matrix whose entries are smooth functions in the variable t.

(c) Show that the Wronskian $W(y_1, \ldots, y_n)$ n solutions $y_1(x), \ldots, y_n(x)$ of

$$\frac{d^n}{dx^n}y + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{7}$$

satisfies

$$\frac{d}{dx}W(y_1, \ldots, y_n) + a_{n-1}(x)W(y_1, \ldots, y_n) = 0 \tag{8}$$

[You can use (b) above, or do it directly using the reasoning for (b).]