MATH 240 ASSIGNMENT 9, SPRING 2015

Due in class on Friday, April 10, 2015

Part 0. Notes on §7.5 and §7.8.

0.1. The theorem below gives the relation between Jordan forms and general solutions of homogeneous system of first order linear ODE's. (Please compare it with Theorem 7.5.4 in the textbook DELA.) In other words the problem of solving a homogeneous system $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ of linear ODE's is reduced to a problem of linear algebra.

Theorem (Jordan form and systems of first order linear ODE) Let A be an $n \times n$ matrix and let

$$\mathbf{v}_{1,1},\ldots,\mathbf{v}_{1,d_1};\mathbf{v}_{2,1},\ldots,\mathbf{v}_{1,d_2};\ldots;\mathbf{v}_{k,1},\ldots,\mathbf{v}_{1,d_k}$$

be a $\mathbb{C}$-basis of $\mathbb{C}^n$ group in k blocks, such that

$$\begin{aligned}(A-\lambda_j)(\mathbf{v}_{j,1}) &= 0\\ (A-\lambda_j)(\mathbf{v}_{j,2}) &= \mathbf{v}_{j,1}\\ (A-\lambda_j)(\mathbf{v}_{j,3}) &= \mathbf{v}_{j,2}\\ &\cdots\\ (A-\lambda_i)(\mathbf{v}_{j,d_j}) &= \mathbf{v}_{j,d_{i-1}}\end{aligned} \tag{1}$$

for each $j = 1,\ldots,k$, where $\lambda_1,\lambda_2,\ldots,\lambda_k$ are complex numbers, not necessarily distinct (consequently $\pm(\lambda-\lambda_1)^{d_1}\cdots(\lambda-\lambda_k)^{d_k}$ is the characteristic polynomial of A.) Then

$$\begin{aligned}\mathbf{x}_{j,1}(t) &:= e^{\lambda_j t}\mathbf{v}_{j,1},\\ \mathbf{x}_{j,2}(t) &:= te^{\lambda_j t}\mathbf{v}_{j,1} + e^{\lambda_j t}\mathbf{v}_{j,2},\\ \mathbf{x}_{j,3}(t) &:= \frac{t^2}{2!}e^{\lambda_j t}\mathbf{v}_{j,1} + te^{\lambda_j t}\mathbf{v}_{j,2} + e^{\lambda_j t}\mathbf{v}_{j,3},\\ &\ \vdots \\ \mathbf{x}_{j,h}(t) &:= \sum_{i=0}^{h-1}\frac{t^i}{i!}e^{\lambda_j t}\mathbf{v}_{j,h-i} \qquad 1 \le h \le d_j,\\ &\ \vdots \\ \mathbf{x}_{j,d_j}(t) &:= \frac{t^{d_j-1}}{(d_j-1)!}e^{\lambda_j t}\mathbf{v}_{j,1} + \frac{t^{d_j-2}}{(d_j-2)!}e^{\lambda_j t}\mathbf{v}_{j,2} + \cdots + te^{\lambda_j t}\mathbf{v}_{j,d_j-1} + e^{\lambda_j t}\mathbf{v}_{j,d_j}\end{aligned} \tag{2}$$

are d_j linear independent solutions of the system of linear first order ODE

$$\frac{d}{dt}\mathbf{x} = A\cdot\mathbf{x} \tag{3}$$

for each $j = 1,\ldots,k$. Moreover

$$\mathbf{x}_{1,1},\ldots,\mathbf{x}_{1,d_1};\mathbf{x}_{2,1},\ldots,\mathbf{x}_{2,d_2};\ldots;\mathbf{x}_{k,1},\ldots,\mathbf{x}_{k,d_k}$$

form a $\mathbb{C}$-basis of the space of all solutions of (3).

0.2. How to solve (3) via matrix exponential.

Step 1. Compute $\det(\lambda \cdot \mathrm{I}_n - A)$ and factor it:

$$\det(\lambda \cdot \mathrm{I}_n - A) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{e_i} \tag{4}$$

where $\lambda_1, \dots, \lambda_r$ are r *distinct* eigenvalues, with multiplicities $e_1, \dots, e_r$. Clearly $e_1 + \cdots + e_r = n$. [This step requires work, and is computationally the hardest. But lets assume that you have all the eigenvalues of A.]

Step 2. Compute the generalized eigenspaces

$$V_i := \mathrm{Ker}(A - \lambda_i \cdot \mathrm{I}_n)^{e_i} \tag{5}$$

of the r eigenvalues $\lambda_1, \dots, \lambda_r$. The output of your computation is a basis $w_{i,1}, \dots, w_{i,e_i}$ of V_i. Let C be the invertible $n \times n$ matrix with $w_{1,1}, \dots, w_{1,e_1}, \cdots, w_{r,1}, \dots, w_{r,e_r}$ as its n columns, i.e.

$$C := [\, w_{1,1}, \dots, w_{1,e_1}; \cdots ; w_{r,1}, \dots, w_{r,e_r} \,] \tag{6}$$

[It is a fact that $\dim(V_i) = e_i$ for each $i = 1, \dots, r$, and these r generalized eigenspaces are linearly *independent*. For each $i = 1, \dots, r$, we have an non-decreasing sequence

$$\mathrm{Ker}(A - \lambda_i \cdot \mathrm{I}_n) \subseteq \mathrm{Ker}(A - \lambda_i \cdot \mathrm{I}_n)^2 \subseteq \cdots \subseteq \mathrm{Ker}(A - \lambda_i \cdot \mathrm{I}_n)^m$$

of subspaces, which stabilizes for $m \geq m_0$ on, for some $m_0 \leq e_i$.]

Step 3. Let Λ be the $n \times n$ diagonal matrix

$$\Lambda := \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_r \end{pmatrix} \tag{7}$$

where each eigenvalue λ_i appears e_i times. Let

$$N := C^{-1} \cdot A \cdot A - \Lambda. \tag{8}$$

Let $e := \max\{e_1, \dots, e_r\}$. The key facts are

(i) $\Lambda \cdot N = N \cdot \Lambda$

(ii) $N^e = 0$

In other words the matrix A is the sum of two commuting $n \times n$ matrices $S := C \cdot \Lambda \cdot C^{-1}$ and $M := C \cdot N \cdot C^{-1}$, where S is diagonalizable and $M^e = 0$. Now we can write down e^{tA}:

$$e^{tA} = C \cdot e^{t\Lambda} \cdot e^{tN} \cdot C^{-1} \tag{9}$$

where

$$e^{t\Lambda} = \begin{pmatrix} e^{\lambda_1 t} & & & & \\ & \ddots & & & \\ & & e^{\lambda_1 t} & & \\ & & & \ddots & \\ & & & & e^{\lambda_r t} \end{pmatrix} \tag{10}$$

and

$$e^{tN} = \mathrm{I}_n + tN + \frac{t^2}{2}N^2 + \frac{t^3}{3!}N^3 + \cdots + \frac{t^{e-1}}{(e-1)!}N^{e-1} \tag{11}$$

We conclude that the general solution of (3) is

$$\mathbf{x}(t) = C \cdot e^{t\Lambda} \cdot e^{tN} \cdot \mathbf{b}, \qquad \mathbf{b} \in \mathbb{C}^n, \tag{12}$$

where $e^{t\Lambda}$ and e^{tN} are given by (10) and (11) respectively. In other words every solution of (3 is a unique linear combination of the n columns of the matrix-valued function $C \cdot e^{t\Lambda} \cdot e^{tN}$ in the variable t.

Part A. (a) Read Part 0 and §§7.5, 7.8, 7.9 of DELA.

(b) Do and write up the answers to following problems from DELA

§7.5 Problems 4, 10, 12.

§7.8 Problems 4, 8, 10.

§7.9 Problem 10, 20.

Part B. Let A be an $n \times n$ matrix in $\mathrm{M}_n(\mathbb{C})$ as in Part 0. Let $\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,d_1}; \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{1,d_2}; \ldots; \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{1,d_k}$ be a $\mathbb{C}$-basis of $\mathbb{C}^n$ group in k blocks, such that satisfying (1).

B1. Show that for each $j = 1, \ldots, k$ we have

$$\begin{aligned} \left(\frac{d}{dt} - A\right) e^{\lambda_j t}\mathbf{v}_{j,1} &= 0 \\ \left(\frac{d}{dt} - A\right) \frac{t^h}{h!} e^{\lambda_j t}\mathbf{v}_{j,1} &= \frac{t^{h-1}}{(h-1)!} e^{\lambda_j t}\mathbf{v}_{j,1} \qquad \text{for all } h \geq 1 \\ \left(\frac{d}{dt} - A\right) e^{\lambda_j t}\mathbf{v}_{j,r} &= -e^{\lambda_j t}\mathbf{v}_{j,r-1} \qquad \text{for all } r = 2, \ldots, d_j \\ \left(\frac{d}{dt} - A\right) \frac{t^h}{h!} e^{\lambda_j t}\mathbf{v}_{j,r} &= -\frac{t^h}{h!} e^{\lambda_j t}\mathbf{v}_{j,r-1} + \frac{t^{h-1}}{(h-1)!} e^{\lambda_j t}\mathbf{v}_{j,r} \qquad \text{for all } h \geq 1,\ r = 2, \ldots, d_j \end{aligned} \tag{13}$$

B2. Use B1 to verify that the functions $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{1,d_1}; \mathbf{x}_{2,1}, \ldots, \mathbf{x}_{2,d_2}; \ldots; \mathbf{x}_{k,1}, \ldots, \mathbf{x}_{k,d_k}$ defined by (2) indeed are solutions of the system (3) of first order linear ODE.

Part C. Extra credit problems.

C1. Use the same notation as in part B. Let C be the invertible $n \times n$ matrix whose n columns are the vectors $\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,d_1}; \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{1,d_2}; \ldots; \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{1,d_k}$.

(a) What is the matrix $C^{-1} \cdot A \cdot C$? (Write down an explicit expression/formula for it.)

(b) Give an explicit expression of the matrix exponential $e^{t\,C^{-1} \cdot A \cdot C}$.

(c) Give an explicit expression of e^{tA} in terms of C and your answer for (b).

C2. (This question has many possible answers.) Suppose you are given an $n \times n$ matrix A and a factorization of its characteristic polynomial. (In particular you have been handed all eigenvalues of A.) According to the theorem on Jordan forms, which you take on faith, there exist a basis $\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,d_1}; \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{1,d_2}; \ldots; \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{1,d_k}$ of $\mathbb{C}^n$ satisfying (1) for a suitable possible integer k and eigenvalues $\lambda_1, \ldots, \lambda_k$. Although the theorem says that k is uniquely determined by A, and the k pairs $(\lambda_1, d_1), \ldots, (\lambda_k, d_k)$ are also uniquely determined up by A up to re-indexing, you are not told what they are. How will you compute a basis

$$\mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,d_1}; \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{1,d_2}; \ldots; \mathbf{v}_{k,1}, \ldots, \mathbf{v}_{1,d_k}$$

with the required properties? Please describe an algorithm (which in principle can be implemented on a computer which has a package installed for solving systems of linear equations.)