3 - Phase plane diagrams for linear systems

Consider the linear homogeneous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{4}$$

Depending on the eigenvalues λ_1, λ_2 of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, various cases arise.

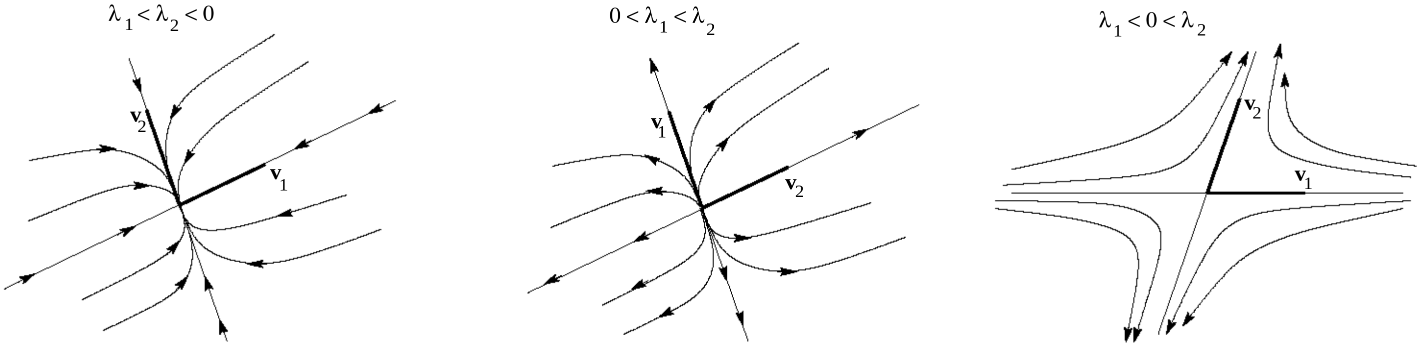
We first assume that the eigenvalues λ_1, λ_2 are real and distinct. Let $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors. The general solution is thus

$$c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 .$$

CASE 1 (stable node): $\lambda_1 < \lambda_2 < 0$. As $t \to +\infty$, all trajectories flow into the origin. The component along $\mathbf{v}_1$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_2$.

CASE 2 (unstable node): $0 < \lambda_1 < \lambda_2$. As $t \to +\infty$, trajectories flow away from the origin, becoming arbitrarily large. For negative times, as $t \to -\infty$, the component along $\mathbf{v}_2$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_1$.

CASE 3 (saddle): $\lambda_1 < 0 < \lambda_2$. The zero solution is unstable. As $t \to +\infty$ the component along $\mathbf{v}_1$ approaches zero, while the component along $\mathbf{v}_2$ becomes arbitrarily large. On the other hand, as $t \to -\infty$, the $\mathbf{v}_1$-component becomes large, while the $\mathbf{v}_2$ component approaches zero.

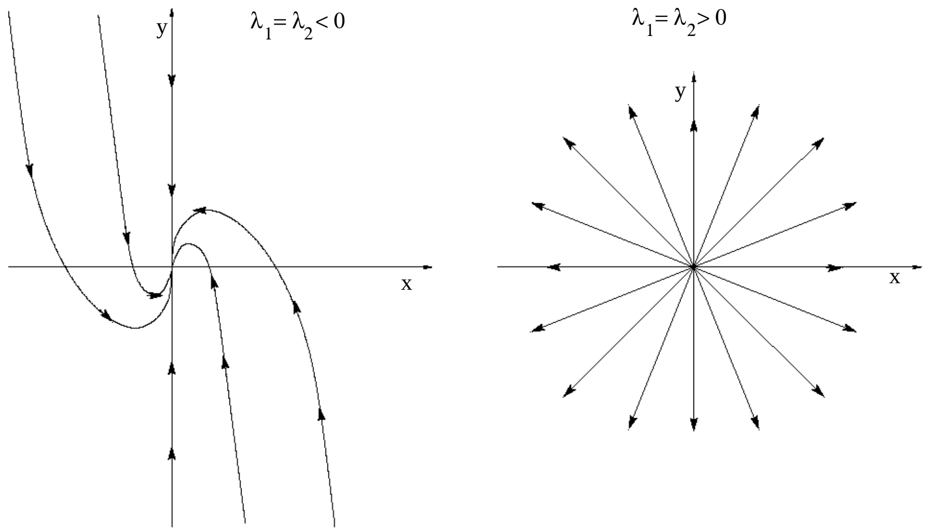


Left: a stable node. Middle: an unstable node. Right: a saddle.

CASE 4 (degenerate node): Assume that the matrix A has a double eigenvalue $\lambda \in \mathbb{R}$.

If $\lambda < 0$ then the origin is a **stable node**. If $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is diagonal, then all trajectories are half lines emanating from the origin. If A is not diagonalizable (only one linearly independent eigenvector $\mathbf{v}_1$ can be found), then trajectories approach the origin tangent to $\mathbf{v}_1$.

If $\lambda > 0$ then the origin is an **unstable node**. The orbits are the same as in the stable case, reversing the time direction.



Left: a stable degenerate node (in the case of only one linearly independent eigenvector).
Right: an unstable degenerate node (in the case of two linearly independent eigenvectors).

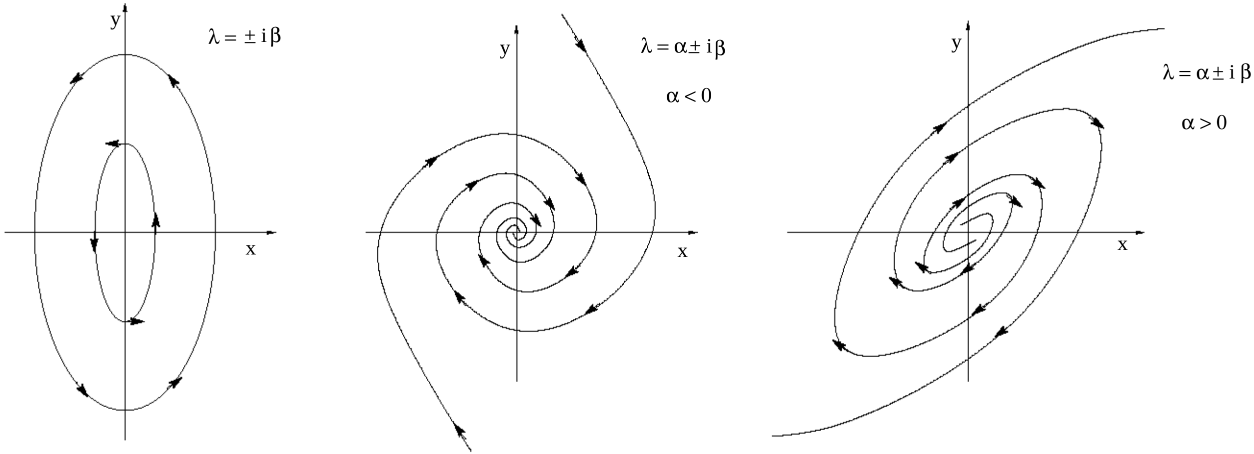
Next, assume that the matrix A has complex eigenvalues: $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$.

CASE 5 (center): If $\alpha = 0$, solutions are periodic. Trajectories are ellipses (or circumferences) centered at the origin.

CASE 6 (stable spiral point): If $\alpha < 0$, trajectories are spirals converging to the origin as

$t \to +\infty$.

CASE 7 (unstable spiral point): If $\alpha > 0$, trajectories are spirals moving away from the origin as time increases.



Left: a center. Middle: a stable spiral point. Right: an unstable spiral point.