Short answers to practice problems set 2

1.

(a) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$: characteristic poly $= \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$.

$\lim\limits_{n\to\infty} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n$ does not exist ; $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$: char. poly $= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$; $\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ is conjugate to $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

$\lim\limits_{n\to\infty} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^n$ does not exist

(c) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$: char. poly $= \lambda^2 - 2\lambda - 1 = (\lambda-1)^2 - 2 = (\lambda - 1 + \sqrt{2})(\lambda - 1 - \sqrt{2})$

$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{pmatrix}$, $\lim\limits_{n\to\infty} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$ does not exist

(d) $\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}$: char. poly $= \lambda^2 - 34 = (\lambda - \sqrt{34})(\lambda + \sqrt{34})$.

$\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}$ is conjugate to $\begin{pmatrix} \sqrt{34} & 0 \\ 0 & -\sqrt{34} \end{pmatrix}$, $\lim\limits_{n\to\infty} \begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}^n$ does not exist

(e) $\begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$: char. poly $= \lambda^2 + \lambda + 2 = (\lambda + \frac{1}{2})^2 + \frac{7}{4} = (\lambda - \frac{1}{2} + \frac{\sqrt{7}i}{2})(\lambda - \frac{1}{2} - \frac{\sqrt{7}i}{2})$

$\begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$ is conjugate to $\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{7}}{2}i & 0 \\ 0 & \frac{1}{2} + \frac{\sqrt{7}}{2}i \end{pmatrix}$,

$\lim\limits_{n\to\infty} \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}^n$ does not exist

2. $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$: char. poly $= (\lambda-1)^2$ $\quad \mathrm{Ker}\left(\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - I_2\right) = \mathrm{Ker}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \mathbb{R}\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{= v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\quad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_1 + v_2$

Let $C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. $\rightsquigarrow$ $AC = C \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ i.e. $A = C \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot C^{-1}$

$\rightsquigarrow$ $A^n = C \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^n \cdot C^{-1} = C \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + n \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \cdot C^{-1} = C \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot C^{-1}$

3. Clearly: (a) $A = A^T$. (b) A is orthogonal $\rightsquigarrow I_4 = A \cdot A^T = A^2$

$\Rightarrow$ The only eigenvalues of A are ± 1

$(A+I_4)\cdot(A-I_4) = 0 = (A-I_4)\cdot(A+I_4) \Rightarrow$ columns of $A+I_4 \in Ker(A-I_4)$

columns of $A-I_4 \in Ker(A+I_4)$

$$A+I_4 = \frac{1}{2}\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix}, \quad A-I_4 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

$\rightsquigarrow \underbrace{\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 3 \\ -1 \\ -1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ 3 \\ -1 \end{pmatrix}}_{v_3}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}}_{v_4}$ is a basis of $\mathbb{R}^4$; v_1, v_2, v_3 are eigenvectors for the eigenvalue -1, v_4 is an eigenvector for the eigenvalue 1

4. $$B = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Observe: $\dim(Ker(B)) = 2$, i.e. 0 is an eigenvalue with multiplicity $(\geq)\ 2$

$B \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 2$ is an eigenvalue

$B \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -2 \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow -2$ is an eigenvalue

$\rightsquigarrow C^{-1} \cdot B \cdot C = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & -2 \end{pmatrix}$ with $C = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$

5. (a) $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (b)(c) eigenvalues of A are: $e^{\frac{\pi i}{4}}, e^{-\frac{\pi i}{4}}$

$\Rightarrow$ There is no invertible $C \in M_2(\mathbb{R})$ such that $C^{-1}AC$ is diagonal

$$Ker\begin{pmatrix} \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \end{pmatrix} = Ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \mathbb{C} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\rightsquigarrow Ker\begin{pmatrix} \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} - \frac{i}{2}\right) & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} - \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \end{pmatrix} = \mathbb{C} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$C^{-1}A \cdot C = \begin{pmatrix} \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{-i}{\sqrt{2}} \end{pmatrix}$

with $C = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 9 \end{pmatrix}$. Let $B = \begin{pmatrix} 1 & 4 \\ -4 & 9 \end{pmatrix}$ char. poly of $B = \lambda^2 - 10\lambda + 25$
$= (\lambda - 5)^2$

$\mathrm{Ker}\left(B - 5 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{Ker}\begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix} = \mathbb{R} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= v_2}$

$B \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{= v_2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} = 4 \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= v_1} + 5 \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{v_2}$ $\rightsquigarrow$ $B \cdot C = C \cdot \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}$
with $C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$B = C \cdot \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix} C^{-1}$ $\quad B^n = C \cdot \left[\begin{pmatrix} 5^n & 0 \\ 0 & 5^n \end{pmatrix} + \begin{pmatrix} 5^{n-1} & 4n \cdot 5^{n-1} \\ 0 & 5^{n-1} \end{pmatrix}\right] C^{-1}$

7. $\left(\frac{d^3}{dx^3} + 3\frac{d^2}{dx^2} + 3\frac{d}{dx} + 1\right) = \left(\frac{d}{dx} + 1\right)^3$

Let $y_{p,1}, y_{p,2}, y_{p,3}$ be a particular solution of $\left(\frac{d}{dx} + 1\right)^3 y = \begin{cases} e^{-x} \\ e^{ix} \\ -1 \end{cases}$
respective (Then $y_p := y_{p,1} + \mathrm{Re}(y_{p,2}) + y_{p,3}$ is a particular sol^n required

Trial solution for $y_{p,1} = a \cdot x^3 e^{-x}$
for $y_{p,2} = b \cdot e^{ix}$
for $y_{p,3} = c$

$\left(\frac{d}{dx} + 1\right)^3 x^3 e^{-x} = 6 \cdot e^{-x}$: take $a = \frac{1}{6}$
$\left(\frac{d}{dx} + 1\right)^3 e^{ix} = (1+i)^3 e^{ix}$: take $b = \frac{1}{8}(1-i)^3 e^{ix}$
$\left(\frac{d}{dx} + 1\right)^3 \cdot 1 = 1$: take $c = -1$

$\rightsquigarrow$ Let $y_p(x) = \frac{1}{6} x^3 e^{-x} + \frac{1}{8} \mathrm{Re}\left[(1-i)^3 e^{ix}\right] - 1$
and $y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + \frac{1}{6} x^3 e^{-x} + \frac{1}{8} \mathrm{Re}\left[(1-i)^3 e^{ix}\right] - 1$
is the requested general solution

8. $\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 5\right)^2 = \left[\left(\frac{d}{dx} + 1\right)^2 + 4\right]^2 = \left(\frac{d}{dx} + 1 + 2i\right)^2 \cdot \left(\frac{d}{dx} + 1 - 2i\right)^2$

Trial solution: $a \cdot x^2 \cdot e^{(-1+2i)x}$

$\left(\frac{d}{dx} + 1 + 2i\right)^2 \cdot \left(\frac{d}{dx} + 1 - 2i\right)^2 \cdot x^2 e^{(-1+2i)x} = \left(\frac{d}{dx} + 1 + 2i\right)^2 \cdot 2 e^{(-1+2i)x}$
$= 2 \cdot (4i)^2 \cdot e^{(-1+2i)x} = -32 \cdot e^{(-1+2i)x}$ $\rightsquigarrow$ take $a = -\frac{1}{32}$

general solution: $y(x) = -\frac{1}{32} x^2 \cdot e^{(-1+2i)x} + c_1 e^{(-1+2i)x} + c_2 x e^{(-1+2i)x}$
$+ c_3 e^{(-1-2i)x} + c_4 x e^{(-1-2i)x}$

9. $y(x) = x \log x \cdot e^{x} + c_1 e^{x} + c_2 x e^{x}$ (variation of parameters)

10. $\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} - 3\right) = \left(\frac{d}{dx} + 3\right)\left(\frac{d}{dx} - 1\right)$

$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 3\right) e^{3ix} = (3i+3)\cdot(3i-1)e^{3ix} = (-9 + 6i - 3)\cdot e^{3ix} = 6(-2+i)\cdot e^{3ix}$

$y_p(x) = \mathrm{Re}\left(\frac{1}{6\cdot 5}(-2-i)\cdot e^{3ix}\right)$

and: general solution is $y(x) = \mathrm{Re}\left(\frac{1}{30}\cdot(-2-i)\cdot e^{3ix}\right) + c_1 e^{x} + c_2 e^{-3x}$

as $x \to \infty$, $\lim_{x\to\infty} e^{-3x} = 0$, while $\mathrm{Re}\left(\frac{1}{30}(-2-i)e^{3ix}\right)$ oscillates

$\lim_{x\to\infty} e^{x} = \infty$

1) must have $c_1 = 0$ (otherwise $\lim_{x\to\infty} y(x) = +\infty$ or $-\infty$ according to whether $c_1 > 0$ or $c_1 < 0$)

2) $\lim_{x\to\infty}\left(\mathrm{Re}\,\frac{1}{30}(-2-i)\cdot e^{3ix}\right)$ does not exist

Ans: $\lim_{x\to\infty} y(x)$ does not exist for every solution.

11. $\left(\frac{d}{dx} + 2i\right)^2\left(\frac{d}{dx} - 2i\right)^2 y = \mathrm{Im}\left(e^{2ix}\right)$

$\left(\frac{d}{dx} + 2i\right)^2\left(\frac{d}{dx} - 2i\right)^2\left(x^2 e^{2ix}\right) = \left(\frac{d}{dx} + 2i\right)^2 \cdot 2\cdot e^{2ix}$

$= 2\cdot(2i)^2\cdot e^{2ix} = -8\,e^{2ix}$

General solution is: $y(x) = -\frac{1}{8}x^2 \sin(2x) + c_1 e^{2ix} + c_2 e^{-2ix}$

$+ c_3 x e^{2ix} + c_4 x e^{-2ix}$

Because of the $-\frac{1}{8}x^2 \sin(2x)$ term,

$y(x)$ is unbounded for all c_1, c_2, c_3, c_4.