## Exponential of a real (3 x 3)-matrix with repeated eigenvalues

If A is a real $(3 \times 3)$-matrix with real eigenvalues $(\lambda_1, \lambda_2, \lambda_2)$ so the characteristic equation

$$p(\lambda) = \det(A - \lambda I) = 0$$

has a single real root $\lambda_1$ and a double real root $\lambda_2$ then the exponential of A is given by

$$e^{tA} = e^{\lambda_2 t} I + te^{\lambda_2 t}(A-\lambda_2 I)$$

$$+ (\lambda_1-\lambda_2)^{-2}(e^{\lambda_1 t} - e^{\lambda_2 t} -(\lambda_1-\lambda_2)te^{\lambda_2 t})(A-\lambda_2 I)^2$$

If A is a real $(3 \times 3)$-matrix with one real eigenvalue $(\lambda, \lambda, \lambda)$ so $\lambda$ is a triple root of the characteristic equation

$$p(\lambda) = \det(A - \lambda I) = 0$$

then the exponential of A is given by

$$e^{tA} = e^{\lambda t}(I + t(A-\lambda I) + \tfrac{1}{2}t^2(A-\lambda I)^2).$$

## Green's Functions

Here are the Green's functions for second order constant coefficient linear differential equation. We find a particular solution $y_p$ to the differential equation $Ly = F$ where $L = P(D) = (D-\lambda_1)(D-\lambda_2)$. Note $y_p(0) = 0$ and $y_p'(0) = 0$.

Case 1. Two distinct real roots $\lambda_1$ and $\lambda_2$.

$$y_p(x) = \frac{1}{\lambda_1-\lambda_2}\int_0^x (e^{\lambda_1(x-t)} - e^{\lambda_2(x-t)})F(t)\,dt$$

$$y_p(x) = \frac{e^{\lambda_1 x}}{\lambda_1-\lambda_2}\int_0^x e^{-\lambda_1 t}F(t)\,dt - \frac{e^{\lambda_2 x}}{\lambda_1-\lambda_2}\int_0^x e^{-\lambda_2 t}F(t)\,dt$$

0's can be changed to any other numbers, a and b. This changes boudary conditions.

Case 2. One real repeated root $\lambda$ so $Ly = P(D)y = (D-\lambda)^2 y$.

$$y_p(x) = \int_0^x e^{\lambda(x-t)}(x-t)F(t)\,dt$$

$$y_p(x) = x\,e^{\lambda x}\int_0^x e^{-\lambda t}F(t)\,dt - e^{\lambda x}\int_0^x te^{-\lambda t}F(t)\,dt$$

0's can be changed to any other numbers, a and b. This changes boundary conditions.

Case 3. $Ly = P(D) = (D-a)^2 + b^2 = D^2 - 2aD + (a^2+b^2)$ $\quad \lambda = a \pm ib$

$$y_p(x) = \frac{1}{b}\int_0^x e^{a(x-t)}(\sin(bx)\cos(bt) - \cos(bx)\sin(bt))\,F(t)\,dt$$

$$y_p(x) = \frac{1}{b}e^{ax}\sin(bx)\int_0^x e^{-at}\cos(bt))\,F(t)\,dt - \frac{1}{b}e^{ax}\cos(bx)\int_0^x e^{-at}\sin(bt))\,F(t)\,dt$$

Again 0's can be changed. This changes boundary conditions

Using a trigonometric identity for sin(x-y) this can be written as

$$y_p(x) = \frac{1}{b}\int_0^x e^{a(x-t)}\sin(b(x-t))\,F(t)\,dt$$

Again 0 can be changed.