Math 241 Homework 10, Fall 2019

Week of November 11; due Monday November 25

Part 1. From the book *Applied PDE* by Haberman.

- Exercise 10.3.7
- Exercise 10.4.7(a), (b).
- Exercise 10.6.3
- Exercise 10.6.9

Part 2. From old final exams.

- Fall 2012 final exam, question 10
- Spring 2013 final exam, question 5
- Fall 2013 final exam, question 8
- Fall 2013 makeup final exam, question 8
- Spring 2015 final exam, question 10
- Fall 2015 final exam, problem 8
- Spring 2016 final exam, question 8
- Fall 2016 final exam, problem 6

Part 3. (extra credit, on Fourier inversion) Let $f(x)$ be a function on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Let

$$F(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{\sqrt{-1}x\omega}\, dx$$

be the Fourier transform of f. Let $x_0 \in \mathbb{R}$, and define a function $g_{x_0}(t)$ by

$$g_{x_0}(t) := \frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0).$$

Assume that there exist a positive number $\delta_0 > 0$ such that

$$\int_0^{\delta_0} \left|\frac{g_{x_0}(t)}{t}\right| dt < \infty.$$

This problem leads you to a proof that

$$f(x_0) = \int_{-\infty}^{\infty} F(\omega)\, e^{\sqrt{-1}\omega x}\, d\omega.$$

under the above assumption on g_{x_0}. Note that this assumption holds if f is differentiable at x_0.

(i) Let $R > \delta$ be positive numbers. (Later we will let $R \to \infty$ and $\delta \to 0$.) Let

$$S_R(x_0) := \int_{-R}^{R} F(\omega)\, e^{\sqrt{-1}\omega x_0}\, d\omega.$$

Show that

$$S_R(x_0) = \frac{1}{\pi}\int_0^{\infty} \frac{\sin(Rt)}{t} \cdot f(x+t)\, dt = \frac{2}{\pi}\int_0^{\infty} \frac{\sin(Rt)}{t} \cdot \frac{f(x+t)+f(x-t)}{2}\, dt.$$

(ii) Use the fact that

$$\int_0^{\infty} \frac{\sin y}{y} dy = \frac{\pi}{2}.$$

to show that

$$S_R(x_0) - f(x_0) = \int_0^{\infty} \frac{\sin(Rt)}{t} \cdot g_{x_0}(t)\, dt.$$

(iii) It is a general fact, call the Riemann–Lebesgue lemma, that for any function $h(x)$ on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |h(x)|\, dx < \infty,$$

we have

$$\lim_{|\omega|\to\infty} \int_{-\infty}^{\infty} h(x) e^{\sqrt{-1}x\omega}\, dx = 0.$$

Show that the Riemann–Lebesgue lemma holds if $h(x)$ is a step function of the form $h(x) = 1$ for $x \in [a, b]$, $h(x) = 0$ for $x \notin [a, b]$, where $a < b$ are real numbers.

[Note: The Riemann–Lebesgue lemma follows from the above special case, for a function $h(x)$ above can be approximated by a linear combination of step functions. Precisely, for every positive number $\epsilon > 0$, there exist a finite linear combination $h_1(x)$ of step functions such that $\int_{-\infty}^{\infty} |h(x) - h_1(x)|\, dx < \epsilon$.]

(iv) Write $S_R(x_0) - f(x_0)$ as a sum of two integrals I_1 and I_2:

$$S_R(x_0) - f(x_0) = \int_0^{\delta} \frac{\sin(Rt)}{t} \cdot g_{x_0}(t)\, dt + \int_{\delta}^{\infty} \frac{\sin(Rt)}{t} \cdot g_{x_0}(t)\, dt =: I_1 + I_2.$$

Use the Riemann–Lebesgue lemma and the assumption on $g_{x_0}(t)$ to show that

$$\lim_{R\to\infty} \big(S_R(x_0) - f(x_0)\big) = 0;$$

i.e. $\int_{-\infty}^{\infty} F(\omega) e^{-\sqrt{-1}x_0\omega}\, d\omega = f(x_0)$.

Note: In the above we have used the equality

$$\int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2}.$$

This formula can be proved in a number of ways. Perhaps the easiest is to use the calculus of residues, an extremely useful tool from complex analysis, also called "complex variables".