Math 241 Homework 2, Fall 2019

Week of September 2 ; due Friday September 13

Reading: §§2.1–2.4 of Haberman

Part 1. From the book *Applied PDE* by Haberman.

- §2.3, exercises 1(b), 2(d), 7
- §2.4, exercise 6

Part 2. From old final exams.

- Fall 2012 final exam, problem 1
- Fall 2015 final exam, problem 1
- Spring 2016 final exam, Q2

Part 3. Extra credit problems.

- Fall 2016 final exam, problem 9. (This is an inhomogeneous linear PDE, with inhomogenous boundary conditions. Try first to find a function $v(x, t)$ which satisfies the the boundary conditions but not necessarily the differential equation. Then rewrite the original PDE for $u(x, t)$ with boundary conditions and initial values to a new PDE for $w(x, t) := u(x, t) - v(x, t)$ with new (now homogeneous) boundary conditions and initial values.

  The method of eigenfunction expansions in the present context means that you try to expand the unknown function $v(x, t)$ as a infinite sum of product functions

  $$a_n(t)\cos(\pi nx) \quad \text{and} \quad b_n(t)\sin(\pi nx),$$

  i.e. the factor involving the x variable is what you got when applying the method of separation of variables, but the factor involving the t variable is "arbitrary".)

- Fall 2014 final exam, Q6. (Note: The real-valued functions $c(x)$, $\rho(x)$, $K_0(x)$ and $f(x)$ are assumed to be smooth on the closed interval $[0, 1]$ and take strictly positive values on $[0, 1]$, but otherwise unspecified. So you will not be able to "solve" this equation by explicit formulas. At present, what you want to do is to figure out the scheme for using separation of variables. You will be able to say something qualitative later, with Sturm–Liouville theory.)

- Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2} \quad t \geq 0,\ 0 \leq x \leq L$$

for homogeneous media and assume that all thermal coefficients are constant with either of the two boundary conditions:

(a) (prescribed boundary value): $u(0,t) = 0 = u(L,t)$ for all $t \geq 0$,

(b) (insulated ends): $\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t)$ for all $t \geq 0$.

For each of the "eigenmode" solution $u(x,t) = \phi(x) \cdot G(t)$ you obtain by the method of separation of variables with boundary condition (a) or (b),

(i) discuss the meaning of "conservation of energy of the entire system represented by this solution" for such an eigenmode solution, and

(ii) determine whether conservation of energy as you formulated in (i) actually holds. Explain the physical reason and meaning of your answer.