Math 241 Homework 3, Fall 2019

Week of September 9 ; due Friday September 20

Reading: §2.5 and §§3.1–3.6 of Haberman

Part 1. From the book *Applied PDE* by Haberman.

- §2.4, exercise 2.4.7 (b)
- §2.5, exercises 2.5.1 (d), 2.5.5(b), 2.5.6 (b)
- §3.3, exercise 3.3.9. Here $f(x)$ is a piecewise smooth function on the interval $[0, L]$, $L > 0$. In the case when $L = \pi$ and $f(x) = 1 + x$ on $[0, \pi]$, express your conclusion explicitly.

Part 2. From old final exams.

- Fall 2012 final exam, problem 2.
- Fall 2013 final exam, problem 3. To justify your answers, either give a sufficient reason why the statement is true, or give a counter-example.
- Spring 2014 final exam, problem 4 (a).

Part 3. Extra credit problems.

A. From old exams

  - Spring 2013 final exam, problem 6. (Notice that part of the boundary condition is that the restriction to the real line of the unknown function $u(x, y)$ is the discontinuous step function with value 1 on $[-1, 1]$ and 0 elsewhere. So the solution $u(x, y)$ might be discontinuous.)
  - Spring 2014 final exam, problem 4 (b).

B. We showed in class that for given a continuous periodic function function $f(\theta)$ on $\mathbb{R}$ with period 2π, the solution of the Laplace equation $\triangle u = 0$ on the closed unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$ such that $u(\cos\theta, \sin\theta) = f(\theta)$ for all θ is given by the formula

$$\begin{aligned} u(r\cos\theta, r\sin\theta) = &\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty} r^n \cos(n\theta)\cdot\frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos(n\phi)\, d\phi \\ &+ \sum_{n=1}^{\infty} r^n \sin(n\theta)\cdot\frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin(n\phi)\, d\phi, \qquad 0 \leq r < 1,\ \theta \in \mathbb{R}. \end{aligned} \tag{1}$$

(a) Show that the formula (1) can be formally rewritten as

$$u(r\cos\theta, r\sin\theta) = \int_0^{2\pi} K(r, \theta-\phi)\, d\phi, \qquad 0 \le r < 1,\ \theta \in \mathbb{R}. \tag{2}$$

where

$$K(r, \theta-\phi) := \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} r^n \cos(n(\theta-\phi)), \quad 0 \le r < 1 \tag{3}$$

(b) Show that

$$K(r, \theta-\phi) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}, \quad 0 \le r < 1. \tag{4}$$

(Hint: Since $r^n \cos(n(\theta-\phi)) = \mathrm{Re}\big[(r\, e^{\sqrt{-1}(\theta-\phi)})^n\big]$, you can compute the infinite series $\sum_{n=1}^{\infty} r^n\, e^{\sqrt{-1}(n(\theta-\phi))}$ first.)