Math 241 Homework 4, Fall 2019

Week of September 16 ; due Friday September 27

Reading: §2.5 and §§3.1–3.6 of Haberman

Part 1. From the book *Applied PDE* by Haberman.

- §3.3, exercise 3.18
- §3.4, exercise 3.4.6
- §3.5, exercise 3.5.6
- §3.6, exercise 3.6.1

Part 2. From old final exams.

- spring 2014 final exam, problem 3
- spring 2016 final exam, Q3
- fall 2016 final exam, problem 4 (Hint: separate the variables.)
- fall 2012 final exam, problem 4 (Hint: separate the variables.)

Summary of term-by-term differentiation of the Fourier series associated to a piece-wise smooth period function $f(x)$ on $\mathbb{R}$ with period 2, as explained in class.

Suppose that $x_1, \dots, x_m$ with $-L < x_1 < \cdots < x_m \leq L$ are non-smooth points of $f(x)$, so that for every non-smooth points differs from exactly one of the x_i's by an integer multiple of $2L$. Define $j_1, \dots, j_m$ as the jumps of $f(x)$ at the non-smooth points, and $j'_1, \dots, j'_m$ be the jumps of the derivative $f'(x)$ of $f(x)$ at the non-smooth points:

$$\begin{aligned} j_k &:= f(x_k+) - f(x_k-) = \lim_{x\to x_k+} f(x) - \lim_{x\to x_k-} f(x) \\ j'_k &:= f'(x_k+) - f'(x_k-) = \lim_{x\to x_k+} f'(x) - \lim_{x\to k_i-} f'(x) \end{aligned} \tag{1}$$

for $k = 1, \dots, m$. Let

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n \, e^{\frac{\pi\sqrt{-1}nx}{L}},$$

i.e. the above infinite series is the Fourier series attached to the given piece-wise smooth periodic function $f(x)$ with period $2L$. Similarly let

$$f'(x) \sim \sum_{n\in\mathbb{Z}} c'_n \, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

and

$$f''(x) \sim \sum_{n\in\mathbb{Z}} c''_n \, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

be the Fourier series attached to the periodic piece-wise continuous functions $f'(x)$ and $f''(x)$. Then

$$\tfrac{\pi\sqrt{-1}n}{L}\, c_n = c'_n + \sum_{k=1}^{m} \tfrac{1}{2L}\, j_k \, e^{\frac{-\pi\sqrt{-1}nx_k}{L}}, \tag{2}$$

and

$$\tfrac{-\pi^2 n^2}{L^2}\, c_n = c'_n + \sum_{k=1}^{m} \tfrac{1}{2L}\, j'_k \, e^{\frac{-\pi\sqrt{-1}nx_k}{L}} + \sum_{k=1}^{m} \tfrac{1}{2L}\, \tfrac{\pi\sqrt{-1}n}{L}\; e^{\frac{-\pi\sqrt{-1}nx_k}{L}}. \tag{3}$$

The reason is that the "correct derivative" of $f'(x)$ in the sense of generalized functions[1] is

$$f'(x) + \sum_{k=1}^{m} j_k \cdot \delta_{x=x_k} \tag{4}$$

because of the jump discontinuity of the x_k's. Here

$$\delta_{x=x_k}$$

denotes the Dirac's δ-function at x_k, so that the integral of $\delta_{x=x_k}$ against any smooth function $\phi(x)$ on $\mathbb{R}$ with bounded support is $\phi(x_k)$. On the other hand, the term-by-term derivative of a Fourier series always represents the "correct derivative", which explains equation (2). The explanation of equation (3) is similar. The last term of (3) comes from the derivatives of $\delta_{x=x_k}$:

$$\int_L^{2L} \delta'_{x=x_k} \cdot e^{\frac{-\pi\sqrt{-1}nx}{L}}\, dx = -\int_L^{2L} \delta_{x=x_k} \left(\tfrac{d}{dx} e^{\frac{-\pi\sqrt{-1}nx}{L}}\right) dx = \tfrac{\pi\sqrt{-1}n}{L}\; e^{\frac{-\pi\sqrt{-1}nx_k}{L}}. \tag{5}$$

The first equality in (5) is the product rule.

Part 3. (Extra credit problem)

(a) Show that the boxed statements on page 116 and page 117 of Harberman's book follow from the statements in the above summary.

(b) Verify the equality (2) for the periodic function $f(x)$ with period 2π such that

$$f(x) = \begin{cases} 1 & \text{if } \; -\pi < x < 0 \\ \cos x & \text{if } \; 0 < x < \pi \end{cases}$$

[1] so that integration by part holds $\int f(x)\phi(x)\, dx$ for every smooth function on $\mathbb{R}$ with bounded support

(c) Assume as before that $f(x)$ is a piece-wise smooth function on $\mathbb{R}$ periodic with period $2L$. Let

$$\sum_{n\in\mathbb{Z}} c_n \, e^{\frac{\pi\sqrt{-1}nx}{L}}$$

be the Fourier series associated to $f(x)$. Does there exist a constant $C > 0$ such that $|c_n| \leq \frac{C}{|n|}$ for all $n \neq 0$? Either fully explain the reason, or give a counter-example.