Math 241 Homework 5, Fall 2019

Weeks of September 23 and September 30; due Monday October 14

Reading: §§4.1–4.5 of Haberman

Part 1. From the book *Applied PDE* by Haberman.

- §4.2, exercise 4.2.2
- §4.4, exercise 4.4.9, 4.4.11

Part 2. From old final exams.

- Fall 2013 make-up final exam, question 7.
- Fall 2016 final exam, problem 7.
- Spring 2013 final exam, question 8. [Hint: You will see another method of solving this inhomogeneous equation later. But it is quicker to first find a product solution of this inhomogeneous PDE (which does not satisfy the boundary condition). Use the solution you find to turn the original problem into a homogeneous PDE with a different boundary condition.]
- Spring 2015 final exam, Q8. [Similar hint to the above: First find a function $u_1(x,t)$ with the same boundary value as required for the unknown function u; you might try to find such a function which is a product of a function in x and a function in t. Then you can turn the original equation to a new equation for $v(x,t) := u(x,t) - u_1(x,t)$, with homogeneous boundary condition and a possibly different initial value at $t = 0$.]
- Spring 2014 final exam, question 6. [You can use the "eigenfunction expansion" method, i.e. use an ODE with homogeneous boundary conditions you obtained by separating the variables, multiply the eigenfunctions thus obtained with general functions in the other variable to form an infinite series as your candidate solution. On the other hand, it is often faster if you can find directly a solution, for instance a product solution, of the inhomogeneous PDE with homogeneous boundary condition (but not the initial condition). Turn the original inhomogeneous PDE with homogeneous boundary conditions and initial conditions into a homogeneous PDE with homogeneous boundary conditions plus initial conditions.]
- (extra credit) Fall 2012 final exam, question 8.

  [Note. 1. The "free end" boundary condition means that the value of the partial derivative with respect to x is 0 at the boundary. In such a non-homogeneous PDE with homogeneous boundary conditions, first you use the method of separation of variables to find the "eigenfunctions" $\phi_n(x)$'s (also called "normal modes"); in this problem they are trig functions, or complex exponentials. Then you write the unknown function $u(x,t)$ as an infinite series, of the form

$$\sum_n g_n(t)\phi_n(x),$$

and the original PDE becomes a family of ODE's for the unknown functions $g_n(t)$, one for each n. Solve the ODE's for $g_n(t)$ to find one particular solution of the non-homogeneous PDE. This is the "method of eigenfunction expansion" in §3.4 of Haberman.

2. If the "source" term of the non-homogeneous wave equation is periodic in x and of the form $f(x)e^{\sqrt{-1}\omega t}$, try to find a special solution of the form $u_0(x,t) = v(x)e^{\sqrt{-1}\omega t}$. You get a non-homogeneous second order ODE for $f(x)$. You can solve this non-homogeneous ODE directly if this ODE has constant coefficients, or expand $v(x)$ as an infinite sum of eigenfunctions $\phi_n(x)$ so that the ODE becomes a family of *algebraic equations*. Denominators in the solution of the resulting non-homogeneous algebraic equations are related to the *resonance* phenomenon, which happens when the denominator is (close to) zero.]