Math 241 Homework 7, Fall 2019

Week of October 14; due Friday October 25

Reading: §7.7 of Haberman

Part 1. From the book *Applied PDE* by Haberman.

- Exercise 7.7.6
- Exercise 7.7.6 (c); here $\alpha(r, \theta)$ denotes "some function of (r, θ), and means that the initial value of the unknown function $u(r, \theta, t)$ at $t = 0$ is given.
- Exercise 7.7.7

Part 2. From old final exams.

- spring 2014 final exam, part (a) of question 7
- fall 2015 final exam, problem 5.

  (Hint: In this problem, to say that $\lambda = 0$ is an eigenvalue means that the eigenfunction ϕ in question is harmonic. So you can use Fourier series to write ϕ as an infinite series. Then examine the boundary condition and translate it into relations between Fourier coefficients.)

  You might have started this problem trying to use Bessel functions. That would work if the question is changed as follows, to a new extra credit problem.

  - (a) Change the boundary condition to $(\nabla\phi, \vec{n}) = 0$ on ∂D, and
  - (b) Ask: For what values of R is $\lambda = 1$ an eigenvalue.
- fall 2013 final exam, question 6

Part 3.

A. (extra credit) The modified Bessel functions $I_n(x)$, $n \in \mathbb{Z}$, are defined by the generating function

$$e^{x(t+t^{-1})/2} = \sum_{n\in\mathbb{Z}} I_n(x) t^n.$$

(a) Find the power series expansion of $I_n(x)$.

(b) Show that

$$\left[(x\tfrac{d}{dx})^2 + (x^2 + \nu^2)\right] I_n = 0.$$

B. Suppose that $y(x)$ on $(0, \infty)$ is a solution of the Bessel equation

$$\Big[(x\tfrac{d}{dx})^2 + (x^2 - \nu^2)\Big]y = 0.$$

Let $u(x) := \sqrt{x}\, y(x)$, defined on $(0, \infty)$.

(a) Show that $u(x)$ satisfies the differential equation

$$\left[\frac{d^2}{dx^2} + \big(1 - \tfrac{\nu^2}{x^2} + \tfrac{1}{4x^2}\big)\right]u = 0.$$

(b) (extra credit) Part (a) shows that for $u(x)$ satisfies a differential equation which is close to the differential equation

$$(\tfrac{d^2}{dx^2} + 1)v(x) = 0.$$

This makes it *plausible* that the $y(x)$ is asymptotically close to a solution of the above equation for sin and cos functions. Can you justify this? What are the difficulties.