Math 241 Homework 8, Fall 2019

Week of October 28; due Monday November 11

Part 1. From the book *Applied PDE* by Haberman.

- Exercise 7.9.2 (c)
- Exercise 7.9.3 (c)
- Exercise 7.9.6

Part 2.

A. Find a solution of the equation

$$\triangle u = 0$$

where the unknown function $u(x, y, z)$ is defined and bounded on the circular cylinder

$$\{x^2 + y^2 \leq a^2,\ 0 \leq z \leq L\},$$

$a, L > 0$ are positive constants, and u satisfies the following boundary conditions

– $u(x, y, 0) = 0 = u(x, y, L)$ for all x, y with $x^2 + y^2 \leq a$,

– $u(x, y) = u_0$ whenever $x^2 + y^2 = a^2$,

where u_0 is a constant. Simplify your answer as much as possible. (Hint: Your solution would involve modified Bessel functions.)

B. The modified Bessel function $I_0(x)$ is a solution of the ordinary differential equation

$$\Big[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - 1\Big]u = 0$$

(a) Let $v(x) = \sqrt{x} I_0(x)$. Show that $v(x)$ is a solution of the differential equation

$$\Big[\frac{d^2}{dx^2} - \big(1 - \frac{1}{4x^2}\big)\Big]v = 0.$$

Note that the above differential equation suggests that $v(x)$ will approximate $A \cdot e^x$ as x goes to ∞, where A is a constant. (In fact $A = \frac{1}{\sqrt{2\pi}}$.)

(b) (extra credit) To improve the approximation $w(x) := e^{-x}v(x)$, show that $w(x)$ satisfies the differential equation

$$\Big[\frac{d^2}{dx^2} + 2\frac{d}{dx} + \frac{1}{4x^2}\Big]w = 0$$

(c) Suppose that

$$w(x) = 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots$$

is a satisfies the differential equation in (b). Substitute $w(x)$ into the differential equation to get

* $c_1 = \frac{1}{8}$,
* $c_2 = \frac{3^2}{2 \cdot 8} c_1$,
* $c_3 = \frac{5^2}{3 \cdot 8} c_2$,

etc. The result in expression is an *asymptotic expansion* of $I_0(x)$.