Math 241 Homework 9, Fall 2019

Week of November 4; due Friday November 15

Part 1. From the book *Applied PDE* by Haberman.

- Exercise 7.10.2 (b)
- Exercise 7.10.12

Part 2. From old final exams.

- Fall 2013 makeup final exam, question 6.

  [Note. Harmonic functions satisfy the following mean value property: suppose that $v(x, y, z)$ is a harmonic function defined on an open domain D in $\mathbb{R}^3$, and $B(x_0, y_0, z_0; a)$ is a closed ball of radius a in D centered at a point (x_0, y_0, z_0). Then $v(x_0, y_0, z_0)$ is equal to the average value of v on the boundary sphere of $B(x_0, y_0, z_0; a)$, and also equal to the average value of v on the ball $B(x_0, y_0, z_0; a)$. Therefore you can do part (b) of this problem without solving the Laplace equation explicitly.]

- Fall 2015 final exam, problem 6.

  [Note: One way to approach this question is to find an ODE satisfied by the total energy $E(t)$. Recall also that by the divergence theorem, the integral of the Laplacian of a function u over a ball is equal to the surface integral of the gradient of this function u over the boundary sphere.]

- Spring 2015 final exam, question 9.

Part 3.

A. (extra credit) The case $d = 3$ in this below was used to get the radial factor in the eigenvalue problem for the Laplacian on a solid ball.

   (i) Verify the following equality of differential operators for every positive integer $d \geq 3$.

$$x^{-(\frac{d}{2}-1)} \left(x\tfrac{d}{dx}\right)^2 x^{\frac{d}{2}-1} = x^{3-d}\tfrac{d}{dx}\left(x^{d-1}\tfrac{d}{dx}\right) + \left(\tfrac{d}{2}-1\right)^2$$

   Note that left hand side of the above formula is the composition of (a) multiplication by $x^{\frac{d}{2}-1}$, (b) the Cauchy–Euler differential operator $x\frac{d}{dx}$ composed with itself, and (c) multiplication by $x^{-(\frac{d}{2}-1)}$.

(ii) Use (i) to write down two linearly independent solutions of the ODE

$$\left[\tfrac{d}{dx}\left(x^{d-1}\tfrac{d}{dx}\right) + \beta x^{d-3} + \lambda x^{d-1}\right]u(x) = 0$$

in terms of Bessel functions, where β is a real number. Note that the left hand side of the above equation is a Sturm–Liouville equation, and λ is considered as an eigenvalue

B. (extra credit) For each natural number n, denote by V_n the set of all homogeneous polynomials in $\mathbb{R}[x, y, z]$ of degree n. Let H_n be the vector subspace of V_n consisting of all *harmonic* homogeneous polynomials $f(x, y, z)$ of degree n, i.e. $\triangle f = 0$.

(i) Show that $\dim_{\mathbb{R}}(V_n) = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$ for each n.

(ii) Deduce from (i) that

$$\dim_{\mathbb{R}}(H_n) \geq \dim_{\mathbb{R}}(V_n) - \dim_{\mathbb{R}}(V_{n-2}) = 2n + 1.$$

(iii) We know that for each natural number n, each of the following functions in spherical coordinates (ρ, ϕ, θ)

$$\rho^n P_n^m(\cos\phi)\cos(m\theta),\ 0 \leq m \leq n, \quad \rho^n P_n^m(\cos\phi)\sin(m\theta),\ 1 \leq m \leq n$$

is harmonic, where $P_n^m(w)$ are the Legendre functions with parameters n, m. Show that each of the above function is a polynomial of degree n in Cartesian coordinates (x, y, z).

(iv) Show that the $2n + 1$ functions in (iii) are linearly independent.

(v) Show that $\dim_{\mathbb{R}}(H_n) = 2n + 1$.

C. (extra credit) This exercise offers an explanation of the fact that the eigenvalues of the Laplace operator on the unit sphere are twice the triangular numbers.

(i) Use fact from Fourier series to show that for every continuous function $f(\phi, \theta)$ on the unit sphere S^2 in $\mathbb{R}^3$ and every positive number $\epsilon > 0$, there exists a finite linear combination

$$Y(\phi, \theta) = \sum_{0\leq n\leq N}\sum_{0\leq m\leq n} A_{mn}P_n^m(\cos\phi)\cos(m\theta) + \sum_{1\leq n\leq N}\sum_{1\leq m\leq n} B_{mn}P_n^m(\cos\phi)\sin(m\theta)$$

such that

$$|f(\phi, \theta) - Y(\phi, \theta)| < \epsilon \quad \forall\, 0 \leq \phi \leq \pi, \forall\, \theta$$

(ii) Conclude that every eigenvalues of the differential equation

$$(\triangle_{S^2} + \lambda)u = 0$$

are of the form $\lambda = n(n + 1)$ for some natural number n.